Method of Lyapunov Functions for Systems with Higher-order Sliding Modes

A. E. Polyakov* and A. S. Poznyak**

* Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
** The Research and Advanced Studies Center of the National Polytechnic Institute (CINVESTAV), Mexico City, Mexico

Received April 18, 2008

Abstract—For the control systems with higher-order sliding modes, a method was proposed to construct the Lyapunov functions on the basis of the method of characteristics for solution of a special first-order partial derivative equation. Its successful solution enables one to generate the Lyapunov function which proves that the convergence time is finite and estimates explicitly the time of reaching the sliding mode.

DOI: 10.1134/S0005117911050043

1. INTRODUCTION

At the end of the XIX century, A.M. Lyapunov proposed a simple approach to the problem of studying stability of the equilibrium of the system of ordinary differential equations (ODE). Its concept relies on the generalized notion of "energy" enabling one to analyze a system for stability or instability using only the information about the right-hand side and doing without the exact solution. At first, these methods were used only for the ODE’s with the right-hand sides that are continuous in both time and state variables. Later on, some authors [1, 2] studied a wider class of differential equations where the right-hand sides are discontinuous in these variables. Such equations are now considered as differential inclusions. They also include the so-called sliding mode systems which invite attention of the researchers over the last four decades [3–6].

The finite time of reaching the sliding surface is one of the most important distinctions of the sliding mode systems. In this case, the corresponding Lyapunov functions may be nonsmooth. For example, in [2] for the first-order sliding systems the simplest of which is given by

\[ \dot{x}(t) = -r \text{sgn}[x(t)], \quad r > 0, \]  

\[ \text{sgn}[x] := \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0 \\
\in [-1, 1] & \text{if } x = 0, 
\end{cases} \]  

(1.1)  

(1.2)

the Lyapunov function is \( V(x) = |x| \), and in virtue of system (1.1), for \( x \neq 0 \) its complete derivative satisfies

\[ \dot{V}(x(t)) = -r. \]

Whence it follows that \( 0 \leq V(x(t)) = V(x(0)) - rt \) and the reach time \( t_{\text{reach}} = V(x(0))/r \).

For the second-order sliding systems (see [3]) such as

\[ \ddot{x}(t) = -r_1 \text{sgn}[x(t)] - r_2 \text{sgn}[\dot{x}(t)], \quad r_1 > r_2 > 0, \]  

(1.3)
the proof of convergence in finite time and the estimate of this time were obtained mostly using
the methods of differential geometry on plane [3] which nobody was able to generalize to the vector
case. Later on, the Lyapunov function
\[ V(x, \dot{x}) = r_1 |x| + \dot{x}^2/2, \tag{1.4} \]
which can guarantee only the asymptotic stability because \( \dot{V}(x(t), \dot{x}(t)) \leq 0 \) was established for
system (1.3) [7]. The existing methods of studying the higher-order sliding systems are based
on the uniformity principle [8–10] according to which the asymptotically stable uniform system
converges in a finite time. Within the limits of this approach, however, it is impossible
to estimate the time of reaching the sliding mode. Therefore, not a single method was proposed enabling one
to prove system convergence in a finite time and estimating this time. The present paper is devoted
to extending the method of Lyapunov functions to the higher-order sliding mode systems which
enables one to resolve the aforementioned problems.

2. METHOD TO GENERATE THE LYAPUNOV FUNCTIONS
WITH FINITE CONVERGENCE TIME

2.1. System Description

Let us consider a control system given by
\[
\begin{aligned}
\dot{x} &= g(x, y) \\
\dot{y} &= a(x, y) + b(x, y)u + f(t, x, y),
\end{aligned}
\tag{2.1}
\]
where \( x \in \mathbb{R}^k \) and \( y \in \mathbb{R}^n \) are the components of the system state vector, \( g : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k \)
and \( a : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n \) are the smooth vector functions, \( u \in \mathbb{R}^m \) is the vector of control inputs,
\( b : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^{n \times m} \) is the matrix of feedback gains, and \( f : \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n \) is an unknown
bounded function, that is,
\[ |f_j(t, x, y)| \leq C_j \quad \forall x \in \mathbb{R}^k, \quad \forall y \in \mathbb{R}^n, \quad \forall t \geq 0, \quad j = 1, n. \tag{2.2} \]
It is assumed that the stabilizing control \( u \), which can be discontinuous,
\[ u = \bar{u}(x, y) \tag{2.3} \]
has already been constructed and it is required to prove with the use of the method of Lyapunov
function that the zero solution \((0, 0)\) of system (2.1) with control (2.3) is stable simply asymptoti-
cally or with a finite time of convergence.

2.2. Generalization of the Zubov Method

Let \( V(x, y) \) be a continuous function and its derivative along the trajectories of system (2.1)
almost everywhere can be estimated as
\[
\begin{aligned}
\dot{V} &= \langle \nabla_x V, g(x, y) \rangle + \langle \nabla_y V, a(x, y) + b(x, y)\bar{u}(x, y) + f(t, x, y) \rangle \\
&\leq \langle \nabla_x V, g(x, y) \rangle + \langle \nabla_y V, a(x, y) + b(x, y)\bar{u}(x, y) \rangle + \langle |\nabla_y V|, C \rangle,
\end{aligned}
\tag{2.4}
\]
where \( \langle \cdot, \cdot \rangle \) stands for the scalar product, \( |\nabla_y V| = \left( |\frac{\partial V}{\partial y_1}|, |\frac{\partial V}{\partial y_2}|, \ldots, |\frac{\partial V}{\partial y_n}| \right)^T \) and \( C = (C_1, C_2, \ldots, C_n)^T \). By denoting
\[ h_j(x, y, \gamma_j) := a_j(x, y) + b_j(x, y)\bar{u}(x, y) + \gamma_j, \quad j = 1, n, \]