The Bubnov–Galerkin Method in Control Problems for Bilinear Systems

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Abstract—We suggest to apply the Bubnov–Galerkin method to solving control problems for bilinear systems. We reduce the solution of a control problem to a finite-dimensional system of linear problem of moments. We show a specific example of applying this procedure and give its numerical solution.

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1. INTRODUCTION

A number of applied problems, including optimization of structure and topology of designs [1–4], certain processes in quantum mechanical systems, ecology, medicine, mathematical economics, engineering and so on [5], can be mathematically modeled by bilinear systems, i.e., systems whose state equations are linear with respect to both functions in question. Bilinear systems are the simplest nonlinear systems that describe a number of real world processes in very different fields of science that would be incorrectly modeled in the framework of linear theory.

The notion of a bilinear system has been introduced to control theory in the 1960s, and since then numerous works have appeared on this topic. A relatively comprehensive list of references can be found in the book [5]. Other control problems for bilinear systems described by ordinary differential equations can be found, e.g., in [6, 7]; partial differential equations, in [8–13].

At present, several exact and approximate methods for solving control problems for bilinear systems have been developed. To solve control problems for bilinear systems, the work [5] systematically uses the theory of matrix Lie groups. The same method has been applied in [8] to prove full controllability by a bilinear control of bending oscillations of plates when the control in question depends on all independent variables. In [13], the required control function occurs in the coefficients of the state function of Schrödinger equation, while in [9, 12] it occurs in that of the first derivative of the state function of the wave equation. An interesting control problem for the coefficient of a Korteweg–de Vries equation which is nonlinear in the state function but linear in the control function has been considered in [11]. In [1], classical variational calculus methods are used to study different problems of the theory of elasticity, when state equations are linear with respect to the state function and nonlinear with respect to the control function. In the studies of structural and typological optimization problems, researchers often employ the method of finite elements together with the Fourier method of separating variables [2–4].

In the present work, we propose a novel approximate method for solving control problems for bilinear equations which is mathematically founded on the Bubnov–Galerkin procedure [14]. We demonstrate this approach with an important example where the required control function does not depend explicitly on one of the independent variables. In such situations we propose to apply the generalized Butkovsky’s finite control method to find the control in question [15].
2. THE BUBNOV–GALERKIN PROCEDURE FOR BILINEAR CONTROL SYSTEMS

The control problem for a bilinear partial differential system usually requires one to optimize a given functional \( \kappa[u] \) by choosing a function \( u \) from a given set \( U \) of admissible controls under differential constraints

\[
D_u[w] = N(x,t), \quad x \in \Omega \subset \mathbb{R}^3, \quad t > 0.
\]  

A solution of (1) satisfies given linear boundary conditions

\[
B[w] = w_0(t), \quad x \in \partial\Omega, \quad t > 0,
\]  

and certain initial conditions. Here \( D_u[w] \) is a differential operator defined in region \( \Omega \times \mathbb{R}^+ \) and containing the product of state function \( w \) and control function \( u \) or their derivatives, \( B[\cdot] \) is a given linear operator defined in the region \( \partial\Omega \times \mathbb{R}^+ \). As usual, \( \partial\Omega \) denotes the boundary of region \( \Omega \), \( \mathbf{n} \) is the vector of its external normal. Examples of operator \( D_u[\cdot] \) can be found, for instance, in [1, 4, 5, 8–13].

The purpose of a control problem can be to provide for solutions of the boundary problem (1) and (2) with required final conditions for fixed \( t = T \). Final conditions are often assumed to be equal to zero.

In the present work, we propose to use the Bubnov–Galerkin procedure [14] to solve this control problem. If we are able to construct a system of linear independent basis (approximating) functions \( \{ \varphi_k(x,t) \}_{k=0}^n \) for the boundary value problem (1), (2) then the residue obtained by substituting approximate solutions \( w_n(x,t) = \varphi_0(x,t) + \sum_{k=1}^n \alpha_k \varphi_k(x,t) \) into Eq. (1) will be

\[
R_n(x,t) \equiv D_u[w_n] - N(x,t), \quad x \in \overline{\Omega}, \quad t > 0.
\]  

According to the Bubnov–Galerkin method, the coefficients \( \alpha_k \) are determined from orthogonality conditions on basis functions \( \{ \varphi_k(x,t) \}_{k=0}^n \) to the residue (4) [14]:

\[
\int_0^T \int_\Omega R_n(x,t) \varphi_k(x,t) \, dx \, dt = 0, \quad k = 1, 2, \ldots, n.
\]  

If for some \( n_0 \in \mathbb{N} \) the residue (4) is identically zero, \( R_n(x,t) \equiv 0 \), then the corresponding approximation \( w_{n_0}(x,t) \) (3) will be an exact solution of the boundary problem (1), (2). Otherwise, increasing the number \( n \) of the terms in (3), we can approximate the solution in question up to a given accuracy. Then the limit case \( w_\infty(x,t) \) will be an exact solution of problem (1), (2).

After we find coefficients \( \alpha_k \) from the system of linear algebraic Eqs. (5) and substitute them into the approximate solution (3), we take into account that at time moment \( T \) given final conditions must hold, for finding the function in question we get a system of constraints of the form

\[
\int_0^T \int_\Omega u \mathcal{K}_k(x,t) \, dx \, dt = \mathcal{M}_k, \quad k = 1, 2, \ldots, n,
\]  

where kernels \( \mathcal{K}_k(x,t) \) and constants \( \mathcal{M}_k \) depend on the parameters of system (1), (2).