Quatertion Regularization in Celestial Mechanics, Astrodynamics, and Trajectory Motion Control. III

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Received April 22, 2013

Abstract—The present paper1 analyzes the basic problems arising in the solution of problems of the optimum control of spacecraft (SC) trajectory motion (including the Lyapunov instability of solutions of conjugate equations) using the principle of the maximum. The use of quaternion models of astrodynamics is shown to allow: (1) the elimination of singular points in the differential phase and conjugate equations and in their partial analytical solutions; (2) construction of the first integrals of the new quaternion; (3) a considerable decrease of the dimensions of systems of differential equations of boundary value optimization problems with their simultaneous simplification by using the new quaternion variables related with quaternion constants of motion by rotation transformations; (4) construction of general solutions of differential equations for phase and conjugate variables on the sections of SC passive motion in the simplest and most convenient form, which is important for the solution of optimum pulse SC transfers; (5) the extension of the possibilities of the analytical investigation of differential equations of boundary value problems with the purpose of identifying the basic laws of optimum control and motion of SC; (6) improvement of the computational stability of the solution of boundary value problems; (7) a decrease in the required volume of computation.

DOI: 10.1134/S0010952515050044

1. THE FORMULATION OF THE PROBLEM OF OPTIMUM CONTROL OF SC CENTER OF MASS

The motion of the center of mass of SC (the material point $B$ of variable masses) will be considered in the coordinate system $OX_1X_2X_3(X)$ with origin at the center of attraction $O$. The coordinate axes of this coordinate system are parallel to the axes of the inertial coordinate system. The controlled motion of the center of mass of the spacecraft (SC) in the central Newtonian field of gravity is described by a differential vector equation of the second order \[ d^2\mathbf{r}/dt^2 + fM\mathbf{r}^{-3}\mathbf{r} = \mathbf{p} \] (1.1) or by a system of two vector differential equations of the first order \[
\begin{align*}
\frac{dr}{dt} &= \mathbf{v}, \\
\frac{dv}{dt} &= -fM\mathbf{r}^{-3}\mathbf{r} + \mathbf{p}.
\end{align*}
\] (1.2)
where $\mathbf{r}$ and $\mathbf{v}$ are the radius vector and the vector of velocity of SC center of mass in the coordinate system $X$, $f = |f|$ is the gravitational constant, $M$ is the mass of an attracting body, $m = m(t)$ is the SC mass, and $\mathbf{p}$ is the vector of acceleration of SC center of mass due to engine thrust, accepted as a control, $T = Te$ is the thrust vector, $T$ and $e$ are the magnitude and unit vector of the thrust direction.

In space flight mechanics, the following problem of optimum control of the motion of SC center of mass in a Newtonian gravitational field is of fundamental importance: it is required to determine the restricted-in-magnitude control $\mathbf{p}$:

\[ 0 \leq p \leq p_{\max} < \infty, \quad p = |\mathbf{p}|, \] (1.3)

which transfers the SC, the motion of the center of mass of which is described by equations (1.1) or (1.2), from the given initial state

\[ \mathbf{r}(t_0) = \mathbf{r}(0) = \mathbf{r}^0, \quad \mathbf{v}(t_0) = \mathbf{v}(0) = \mathbf{v}^0, \] (1.4)

to the final state

\[ \mathbf{r}(t_f) = \mathbf{r}^*, \quad \mathbf{v}(t_f) = \mathbf{v}^* \] (1.5)
or to a state that belongs to some movable diversity, in the general case, and minimizes the functional

\[ J = \int_{t_0}^{t_f} (\alpha_1 + \alpha_2 p^2(t))dt, \quad \alpha_1, \alpha_2 = \text{const} \geq 0 \] (1.6)
or the functional
\[ J = \int_{0}^{t_{f}} (\alpha_{1} + \alpha_{2} p(t)) dt = \alpha_{1} t_{f} + \alpha_{2} Q(t_{f}), \]  
(1.7)
where \( \alpha_{1}, \alpha_{2} \) are weighting coefficients of the functional. The time \( t_{f} \) of controlled motion is supposed to be nonspecified.

Functional (1.6) for \( \alpha_{1} = 0, \alpha_{2} = 1 \) is used in problems of the mechanics of space flight with low-thrust engines [6], and functional (1.7) is used, for the same values of constants \( \alpha_{1} \) and \( \alpha_{2} \), in problems with high-thrust engines [5]. Functional (1.6) characterizes energy expenses for SC transfer from an initial to final state and the time spent for this transfer. Functional (1.7) characterizes the total (in some proportion) SC time and characteristic velocity for performing controlled motion. For \( \alpha_{1} = 0, \alpha_{2} = 1 \) the minimum of functional (1.7) implies the minimum of characteristic velocity \( Q \). For \( \alpha_{2} = 0 \) functionals (1.6), (1.7) transfer to the functional \( J = t_{f} \), and the formulated problem represents a high-speed response problem in this case. Note that the solution of the problem of optimum control of motion of SC center of mass for functional (1.7) is much more difficult than the solution of a similar problem for functional (1.6), because of the nonanalytical character of the integrand function in equation (1.7), which results in laborious calculations in determining optimum control.

The formulated problem will be considered using Pontryagin’s maximum principle. We introduce the additional variable \( g \), which satisfies, when minimizing functional (1.6), the differential equation \( \dot{g} = \alpha_{1} + \alpha_{2} p^{2}(t) \) and the initial condition \( g(0) = 0 \); or, when minimizing functional (1.7), the differential equation \( \dot{g} = \alpha_{1} + \alpha_{2} p(t) \) and the initial condition \( g(0) = 0 \). We introduce conjugate vector variables \( \psi_{r} \) and \( \psi_{v} \), corresponding to vector phase variables \( r \) and \( v \), and the scalar conjugate variable \( \psi_{0} \), corresponding to the scalar phase variable \( g \).

The Hamilton–Pontryagin function will be as follows:
\[
H = \psi_{0} \sigma + \psi_{r} \cdot v - fMr^{-3} \psi_{v} \cdot r + \psi_{v} \cdot p, \tag{1.18}
\]
where for functional (1.6)
\[
\sigma = \alpha_{1} + \alpha_{2} p^{2}, \quad \alpha_{1} \geq 0, \quad \alpha_{2} \geq 0, \tag{1.19}
\]
and for functional (1.7)
\[
\sigma = \alpha_{1} + \alpha_{2} |p|, \quad \alpha_{1} \geq 0, \quad \alpha_{2} \geq 0. \tag{1.20}
\]

The system of equations for the conjugate variables has a well-known form:
\[
\begin{align*}
\frac{d\psi_{r}}{dt} &= -\psi_{r}, \\
\frac{d\psi_{v}}{dt} &= fMr^{-3}\psi_{v} - 3fMr^{-5}(\psi_{v} \cdot r)r, \\
\frac{dp}{dt} &= 0.
\end{align*} \tag{1.11}
\]

In accordance with the principle of the maximum, \( \psi(0) \leq 0 \), therefore, in virtue of equation (1.12) and the homogeneity of function \( H \) in conjugate variables, one can choose any \( \psi_{0}(t) = \text{const} < 0 \) by appropriate redetermination of the other variables. Further on, in the expression (1.8) for the function \( H \), the multiplier \( \psi_{0} \) is supposed to be equal to \(-1\).

The optimum control \( p^{0} \) found from the condition of maximum of function \( H \) defined by relations (1.8)—(1.10), with respect to variable \( p \) with regard to constraint (1.3), has the form:
\[
p^{0} = p^{0} \psi_{v} \left( \psi_{r} \right). \tag{1.13}
\]

Here, in for the minimization of the functional (1.6) in the case of \( \alpha_{2} > 0 \),
\[
p^{0} = \begin{cases} (2\alpha_{2})^{-1} |\psi_{v}|, & \text{if } (2\alpha_{2})^{-1} |\psi_{v}| \leq p_{\text{max}}, \\
p_{\text{max}}, & \text{if } (2\alpha_{2})^{-1} |\psi_{v}| > p_{\text{max}}, \end{cases} \tag{1.14}
\]
and in the case of \( \alpha_{2} = 0 \)
\[
p^{0} = p_{\text{max}}. \tag{1.15}
\]

In the minimization of the functional (1.7),
\[
p^{0} = \begin{cases} p_{\text{max}}, & \text{if } |\psi_{v}| - \alpha_{2} \geq 0, \\
0, & \text{if } |\psi_{v}| - \alpha_{2} < 0, \end{cases} \tag{1.16}
\]

Note that among the control modes there exist: (1) a mode corresponding to the maximum value of acceleration \( p \); (2) a mode corresponding to the minimum (zero) value of acceleration, as well as (3) a particular mode that can arise when meeting the third of the conditions in (1.16).

Equations (1.2), (1.11) of the considered boundary value problem have, for the optimum control and optimum trajectory (more precisely, for any \( p \mid \psi_{v} \)), in particular, when meeting the necessary optimality conditions (1.13)—(1.16), which take into account the restriction imposed on the control magnitude), the vector (1.17) and scalar (1.18) first integrals [7—9]:
\[
((dr/dt) \times \psi_{v} + (d\psi_{r}/dt) \times r)|_{p = p^{0}} = \text{const}, \tag{1.17}
\]
\[
H(r, v, \psi_{r}, \psi_{v}, p^{0}(\psi_{v})) = 0. \tag{1.18}
\]

\(^2\) Here and hereafter the control satisfying the necessary optimality conditions (Pontryagin’s maximum principle) is called the optimum control, and the optimum trajectory is the trajectory that meets this control.