SHORT COMMUNICATIONS

On Functional-Differential Equations with Discontinuous Right-Hand Side

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Abstract—We consider unique determination and right uniqueness issues for solutions, satisfying the one-sided Lipschitz condition, of functional-differential equations with discontinuous right-hand side.

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1. IMPLICIT FORM OF FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDE

Let \( R^n \) be the \( n \)-dimensional vector space with the Euclidean norm \( \| \cdot \| \), and let \( C_\tau \) be the space of all continuous functions \( \psi(\cdot) \) defined on the closed interval \([-\tau, 0]\), \( \tau > 0 \), and ranging in \( R^n \) with the ordinary sup-norm \( \| \psi(\cdot) \| = \sup_{-\tau \leq \theta \leq 0} \| \psi(\theta) \| \). Suppose that a function \( f : \mathbb{R}^1 \times C_\tau \to \mathbb{R}^n \) is continuous everywhere except for manifolds of the form

\[
S_j = \{ (t, \psi(\cdot)) : W_j(t, \psi(0), \psi(\cdot)) = 0 \} \subset \mathbb{R}^1 \times C_\tau, \quad j = 1, \ldots, m,
\]

where the \( W_j : \mathbb{R}^n \times C_\tau \to \mathbb{R}^1 \) are invariantly differentiable functionals. Consider the functional-differential equation

\[
\dot{x} = f(t, \psi(\cdot)).
\]

A solution of Eq. (1) is defined as a solution of the functional-differential inclusion

\[
\dot{x} \in F(t, \psi(\cdot)),
\]

where \( F \) is the convex hull of all limit values of the function \( f \) at each point \((t, \psi(\cdot))\).

For an arbitrary function \( \psi(\cdot) \in C_\tau \) and for a number \( \Delta > 0 \), by \( E_\Delta(\psi(\cdot)) \) we denote the set of all continuous extensions of the function \( \psi(\cdot) \) to the interval \([-\tau, \Delta]\).

We introduce the following notions [1, p. 44].

**Definition 1.** A functional \( V : C_\tau \to \mathbb{R}^1 \) is said to have an *invariant derivative* \( \partial_\psi V \) at a point \( \psi(\cdot) \in C_\tau \) if for each \( \Psi(\cdot) \in E_\Delta(\psi(\cdot)) \) the function \( Y_\psi(\xi) = V(\Psi(\cdot)) \), where \( \xi \in [0, \Delta] \) and \( \Psi_\xi(\theta) = \Psi(\xi + \theta), -\tau \leq \theta \leq 0 \), has a finite right derivative \( \partial Y_\psi / \partial \xi_{\xi=+0} \) at zero invariant with respect to the functions \( \Psi(\cdot) \in E_\Delta(\psi(\cdot)) \); i.e., the value of the right derivative at zero is the same for all \( \Psi(\cdot) \in E_\Delta(\psi(\cdot)) \).

**Definition 2.** A functional \( W : R^n \times C_\tau \to R \) is said to be *invariantly differentiable* at a point \( p = (x, \psi(\cdot)) \in R^n \times C_\tau \) if there exist finite values of \( \nabla_x W \) and \( \partial_\psi W \) at that point and the relation

\[
W(x + z, \Psi(\cdot)) - W(x, \psi(\cdot)) = \langle \nabla_x W[p], z \rangle + \partial_\psi W[p]\xi + o\left(\sqrt{\|z\|^2 + \xi^2}\right)
\]

holds for any function \( \Psi(\cdot) \in E_\Delta(\psi(\cdot)) \), \( z \in R^n \) and \( \xi \in [0, \Delta] \); moreover, \( o(\cdot) \) depends on the choice of the function \( \Psi(\cdot) \in E_\Delta(\psi(\cdot)) \). (Here \( \nabla_x W \) is the gradient of the functional \( W \) with respect to the variable \( x \), and \( \langle \cdot, \cdot \rangle \) stands for the inner product.)
We set \( S \equiv \bigcup_i S_i \). We represent the complement of the set \( S \) in the form of the union of sets \( \Omega_i \) for each of which the functionals \( W_j \neq 0 \) preserve signs on any cross-section \( \{ \psi(\cdot) : (t, \psi(\cdot)) \in \Omega \} \) of the set \( \Omega \), by the point \( t = \text{const} \). We assume that, for each \( \Omega_i \) and for its arbitrary boundary point \((t, \psi(\cdot)) \in S\), there exists a finite limit \( f(t, \psi(\cdot)) \) of the function \( f(t', \psi'(\cdot)) \) provided that \((t', \psi'(\cdot)) \in \Omega_i \). Note that, at each point \((t, \psi(\cdot))\), there are finitely many limits since the set of domains \( \Omega_i \) is finite. We have \( f(t, \psi(\cdot)) = f(t, \psi(\cdot)) \) at a point of continuity.

A function \( f \) with the above-mentioned properties is said to be \textit{piecewise continuous}.

For any vector \( z \in \mathbb{R}^n \) and number \( h \in [0, \tau) \), we introduce the function

\[
\psi^z(\theta) = \begin{cases} 
\psi(h + \theta) & \text{if } -\tau \leq \theta \leq -h \\
\psi(0) + (h + \theta)z & \text{if } -h \leq \theta \leq 0.
\end{cases}
\]

By \( f(t, \psi(\cdot); z) \) we denote the limit of the function \( f(t, \psi^z(\cdot)) \) as \( h \to +0 \) and set \( \Omega = \bigcup_i \Omega_i \). If \((t, \psi(\cdot)) \in \Omega \), then \( f(t, \psi(\cdot); z) = f(t, \psi(\cdot)) \) for any vector \( z \in \mathbb{R}^n \), since, in this case, \((t, \psi(\cdot)) \) is a point of continuity of the function \( f(t, \psi(\cdot)) \). Let \( (t, \psi(\cdot)) \in S \). The limit value \( f(t, \psi(\cdot); z) \) is uniquely determined for a piecewise continuous function \( f \) if there exists a \( \delta > 0 \) such that the points \((t, \psi^z(\cdot)) \) belong only to one of the sets \( \Omega_i \) for all \( h \in (0, \delta) \). The latter is valid if, for all \( i \in \mathbb{N} \), \( \{ z : W_i(\psi(0), \psi(z)) = 0 \} \), the functionals \( W_i(\psi^z(0), \psi^z(\cdot)) \) are nonzero and preserve their signs for all \( h \in (0, \delta) \). By using the assumption on the invariant differentiability of the functional \( W_i \), we write out the relation

\[
W_i(\psi(0) + hz, \psi^z(\cdot)) = W_i(\psi(0), \psi(\cdot)) + \langle \nabla W_i[p], z \rangle h + \partial^\circ W_i[p] h + o(h) \sqrt{1 + h^2}
\]

at the point \( p = (\psi(0), \psi(\cdot)) \).

Since \( W_i = 0 \) for \( i \in I(\psi(\cdot)) \), it follows from (3) that

\[
W_i(\psi(0) + hz, \psi^z(\cdot)) / h = \langle \nabla W_i[p], z \rangle + \partial^\circ W_i[p] + o(h) / h,
\]

whence we find that for a sufficiently small \( \delta > 0 \), for all \( h \in (0, \delta) \), and for any \( z \) the sign of \( W_i \) coincides with that of \( p_i(\psi(\cdot), z) \) provided that

\[
p_i(\psi(\cdot), z) \equiv \langle \nabla W_i[p], z \rangle + \partial^\circ W_i[p] \neq 0
\]

for all \( i \in I(\psi(\cdot)) \). Therefore, the value of \( f(t, \psi(\cdot); z) \) is uniquely determined provided that the vector \( z \) satisfies the condition \( p_i(\psi(\cdot), z) \neq 0 \) for all \( i \in I(\psi(\cdot)) \).

Let us consider two cases.

1. Let \( \nabla W_i[p] = 0 \) for some index \( i \in I(\psi(\cdot)) \). Then \( p_i(\psi(\cdot), z) \neq 0 \) only if \( \partial^\circ W_i[p] \neq 0 \). Therefore, it follows from (4) for this index \( i \) that \( W_i = 0 \) and this functional has the sign of \( \partial^\circ W_i[p] \) for sufficiently small \( \delta > 0 \) and for all \( h \in (0, \delta) \) and \( z \in \mathbb{R}^n \).

2. If \( \nabla W_i[p] \neq 0 \), then the equations \( p_i(\psi(\cdot), z) = 0 \) with a fixed \( \psi(\cdot) \) define hyperplane \( K_i(\psi(\cdot)), \) which is parallel to the subspace \( \{ z \in \mathbb{R}^n : \langle \nabla W_i[p], z \rangle = 0 \} \) tangent to the surface \( S_i = \{ x : W_i(x, \psi(\cdot)) = 0 \} \), at the point \( x = \psi(0) \) [for a fixed function \( \psi(\cdot) \)].

We introduce the set of indices \( J_i(\psi(\cdot)) = \{ j \in I(\psi(\cdot)) : \langle \nabla W_i[p], z \rangle = 0 \} \). By taking into account all preceding considerations, we find that it is sufficient for the validity of condition (5) that \( z \) does not belong to \( K_i(\psi(\cdot)) \) for any \( i \in J(\psi(\cdot)) \).

Note some obvious properties of the mapping \( z \to f(t, \psi(\cdot); z) \) for arbitrary fixed \((t, \psi(\cdot))\).

1. The mapping \( z \to f(t, \psi(\cdot); z) \) is defined at all points \( z \) lying outside the set

\[
K(\psi(\cdot)) = \bigcup_i K_i(\psi(\cdot)) = \bigcup_i K_i(\psi(\cdot)) : i \in J(\psi(\cdot)) \}
\]

2. The hyperplanes \( K_i(\psi(\cdot)) \) divide the space \( \mathbb{R}^n \) into open parts \( G^i \) on each of which the function \( z \to f(t, \psi(\cdot); z) \) is constant.