Periodic Solutions of Evolution Equations with Homogeneous Principal Part

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Received January 22, 2003

Abstract—We prove a priori estimates and existence theorems for periodic solutions of evolution equations whose principal part is homogeneous with respect to the phase variable and is nonstationary in time. We outline applications to differential equations of the parabolic type.

DOI: 10.1134/S0012266108080077

In the present paper, we use ideas of nonlinear analysis [1–3] and methods of the theory of mappings of monotone type [4–11]. The novelty in our results is related to a more comprehensive (for example, compared with [5–8, 12–16]) investigation of the dependence of solutions of the Cauchy problem on the initial values and the mappings occurring in the evolution equations. Lemma 2 can be useful, in particular, for justifying the averaging method for nonlinear parabolic equations. A number of assumptions on the structure of the equations to be studied is introduced for the first time and is quite complicated for verification. Such difficulties are also specific for systems of ordinary differential equations [1–3]. The infinite-dimensional case has not been sufficiently studied at all.

Let us introduce the main notation: \( \| x; X \| = \| x \|_X \) is the norm of an element \( x \) in a \( B \)-space \( X \); \( X^* \) is the dual space of \( X \); \( (x, x^* ) \) is the value of a functional \( x^* \) in \( X^* \) on an element \( x \) in \( X \); \( \sigma (X, X^* ) \) and \( \sigma (X^* , X) \) are the weak topologies induced on \( X \) and \( X^* \) by the form \( (x, x^* ) \); \( L^p (T, X) \) (\( 1 \leq p \leq \infty \)) and \( C(T, X) \) are the Banach spaces of Lebesgue–Bochner measurable functions and continuous functions, respectively, on a closed interval \( T \subset \mathbb{R} \) ranging in the space \( X \) [5]; as usual, functions almost coinciding with respect to the Lebesgue measure \( mes_1 \) are identified; the norms in the spaces \( L^p (T, X) \) and \( C(T, X) \) are introduced in a standard way (e.g., see [5, pp. 148, 154 of the Russian translation]); \( C_{w}(T, X) \) is a topological space of weakly continuous functions on the interval \( T \subset \mathbb{R} \) ranging in \( X \); a system of seminorms defining a topology in \( C_{w}(T, X) \) is given by the relation \( \| v \|_l = \| l(v); C(T) \| \), where \( l \in X^* \); \( L^p (T) = L^p (T, \mathbb{R}) \), \( C(T) = C(T, \mathbb{R}) \); all Banach spaces are considered over the field \( \mathbb{R} \) of real numbers; \( \mathbb{R}_+ = [0, \infty) \).

For Banach spaces \( X_1 \) and \( X_2 \), a mapping \( F : X_1 \to X_2 \) is said to be 

- bounded if it takes bounded sets to bounded sets;
- demicontinuous if the function \( v \to (F(v), u^* ) \) is continuous on \( X_1 \) for each \( u^* \in X_2^* \);
- \( m \)-homogeneous if \( F(kx) = k^m F(x) \) (\( x \in X_1 \), \( k > 0 \)).

The notation \( X_1 \to X_2 \) (respectively, \( X_1 \Rightarrow X_2 \)) implies that \( X_1 \) is continuously (respectively, compactly) embedded in \( X_2 \).

1. Let \( V \) be a reflexive separable \( B \)-space with norm \( \| \cdot \| \), let \( H \) be a Hilbert space with norm \( \| \cdot \| \), let \( V \Rightarrow H \), let \( V \) be dense in \( H \), let \( V^* \) be the dual space of \( V \), and let \( \| \cdot \|_*, \) be the norm in \( V^* \). The space \( H \) is identified with its dual \( H^* \), and \( H^* \) is identified with some subspace of \( V^* \). The number \( (v, v^* ) \) stands for both the inner product of elements \( v \) and \( v^* \) in \( H \) and the value of the functional \( v^* \in V^* \) on the element \( v \in V \).

To a closed interval \( T \subset \mathbb{R} \), a number \( p \in (1, \infty) \), and the space \( V \), we assign the Banach spaces \( Y = L^p (T, V) \) and \( Z = L^q (T, V^*) \) [\( q = p/(p-1) \)]. The space \( Y \) (respectively, \( Z \)) can be identified with the dual space of \( Z \) (respectively, \( Y \)) by means of the bilinear form

\[
\langle y, z \rangle = \int_T (y(t), z(t)) dt.
\]
The weak topologies $\sigma(Y, Z)$ and $\sigma(Z, Y)$ on the spaces $Y$ and $Z$, respectively, can be introduced in a standard way. If it is necessary to emphasize the dependence of the spaces $Y$ and $Z$ on $T$, then we denote them by $Y(T)$ and $Z(T)$.

We set $W = \{ y \in Y, y' \in Z \}$. The space $W$ with the norm $\| y \|_W = \| y \|_Y + \| y' \|_Z$ is continuously embedded in the space $C(T, H)$; this permits one to introduce the notion of the value of a function $y$ in $W$ at each point of $T$. If $y \in W$, then the function $t \to |y(t)|^2$ is absolutely continuous on $T$, and $(|y(t)|^2)' = 2(y(t), y'(t))$ almost everywhere [5, p. 177 of the Russian translation].

By $\Phi(T, V, K)$ we denote the set of mappings $F : T \times V \to V^*$ satisfying the following conditions: the mapping $\Phi(\cdot, v) : T \to V^*$ is measurable for any $v \in V$, the mapping $\Phi(t, \cdot) : V \to V^*$ is demicontinuous for almost all $t \in T$, and $\| \Phi(t, v) \|_* \leq K (1 + \| v \|^{p-1})$ with a positive constant $K$.

The function $\Phi(t, y(t))$ is measurable for any measurable function $y : T \to V$. The operator $F(y) = \Phi(\cdot, y(\cdot))$ is a well-defined bounded demicontinuous mapping of $Y$ into $Z$.

Let $F \in \Phi(T, V, K)$, $\zeta \in Z$, and $T = [\alpha, \beta]$. A solution of the differential equation $\zeta = y' + F(y)$ is defined as a function $y$ of the class $W$ such that $\zeta = y' + F(y)$; if, in addition, $y(\alpha) = h$, then $y$ is a solution of the Cauchy problem

$$\zeta = y' + F(t, y), \quad y(\alpha) = h. \quad (1)$$

In this section, we analyze the dependence of solutions of the Cauchy problem on the initial value $h$, the function $\zeta$, and the mapping $F$.

In what follows, $E$ is a reflexive space, $V$ is compactly and densely embedded in the space $E$, $V^*$ and $E$ are continuously embedded in some locally convex topological space, and $W_1 = L^p(T, E)$. The space $W_1$ can be identified with the space $L^p(T, E^*)$. We have the embeddings [7, p. 70] $Y \to W_1$, $W_1^* \to Z$, and $W \to W_1$. A sequence $\zeta_i$ in $Z$ is said to be semiconvergent if it can be represented in the form $\zeta_i = \varphi_i + \psi_i$, where $\varphi_i \to \varphi$ in $Z$ and $\psi_i \to \psi$ in $\sigma(W_1^*, W_1)$; in this case, we write $\zeta_i \to \zeta = \varphi + \psi$.

**Lemma 1.** Let $\| y_i \|_W \leq R$ $(i = 1, 2, \ldots)$, $y_i \to y$ in $\sigma(Y, Z)$, and $\zeta_i \to \zeta$. Then $(y_i, \zeta_i) \to (y, \zeta)$ in $\sigma(L^1(T), L^\infty(T))$.

**Proof.** Let $\zeta_i = \varphi_i + \psi_i$ $(i = 1, 2, \ldots)$, $\varphi_i \to \varphi$ in $Z$, and $\psi_i \to \psi$ in $\sigma(W_1^*, W_1)$. Since $\| y_i \| \leq R$ and $y_i \to y$ in $\sigma(Y, Z)$, we have $y_i \to y$ in $W_1$.

The above-mentioned properties of the sequences $\varphi_i, \psi_i, y_i$ imply the relations

$$\langle 1_\Delta y_i, \varphi_i \rangle \to \langle 1_\Delta y, \varphi \rangle, \quad \langle 1_\Delta y_i, \psi_i \rangle \to \langle 1_\Delta y, \psi \rangle;$$

here $1_\Delta$ is the characteristic function of a measurable set $\Delta \subset T$. This, together with the arbitrary choice of $\Delta$, implies the convergence $(y_i, \zeta_i) \to (y, \zeta)$ in $\sigma(L^1(T), L^\infty(T))$. The proof of the lemma is complete.

Consider the sequence of Cauchy problems

$$\zeta_i = y' + F_i(t, y), \quad y(\alpha) = h_i \quad (1_i)$$

similar to (1); here and throughout the following, $h_i \in H, \zeta_i \in Z, F_i \in \Phi(T, V, K)$ $(i = 0, 1, \ldots)$, $K$ is a constant independent of $i$, and $F_i : Y \to Z$ is the superposition operator induced by $F_i$. We fix a dense subspace $Y_0$ in the space $Y$. The sequence of the Cauchy problems (1) is investigated under the following assumptions.

$$(A_1)$$ If $u \in Y, v \in Y, \| u \|_Y \leq R, \text{ and } \| v \|_Y \leq R,$$ then

$$\langle u - v, F_i(u) - F_i(v) \rangle \geq -c(R, \| u - v \|_{W_1}) \quad (i = 1, 2, \ldots), \quad (2)$$

where $c(R, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function independent of $i$ and $c(R, \xi) = o(\xi)$ as $\xi \to +0$.

$$(A_2)$$ $\zeta_i \to \zeta$ and $h_i \to h_0$ in $\sigma(H, H)$, and $F_i(u) \to F_0(u)$ for all $u \in Y_0$. Without loss of generality, we assume that $F(t, 0) = 0$. [Indeed, the mapping $F_i$ can be replaced by the mapping $F_i - F_i(t, 0).$]