On the Local Solvability of Evolution Equations for the Interface of Two Filtering Fluids in the Class of Analytic Functions

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Abstract—We consider a system of equations that describes the evolution of the interface of two fluids in the two-dimensional problem of their combined filtration in a porous homogeneous medium. This system contains a nonlinear integro-differential equation with a singular integral over an unknown contour together with a Fredholm integral equation of the second kind for the jump of the velocity vector potential on the movable boundary. We prove the unique solvability of such a system on a small time interval for the case in which the initial shape is defined parametrically and the functions describing the dependence of the coordinates of the point on the line on the parameter admit analytic continuations into the complex plane.

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We consider the system of equations obtained in [1] that describes the evolution of the interface of two fluids in the two-dimensional problem of their combined filtration in a porous homogeneous medium. This system contains a nonlinear integro-differential equation with a singular integral over an unknown contour. This equation is close in structure to the Birkhoff equation, which describes the motion of the tangential discontinuity line of the perfect fluid velocity. The existence and uniqueness of the solution of the latter equation on a small time interval was proved in [2] for the case in which the initial discontinuity shape is described parametrically by functions admitting analytic continuations into the complex plane and the solution is sought in the class of such functions.

The system of equations describing the evolution of the fluid interface in our filtration problem is characterized by the presence of an additional linear integral equation for the jump of the vector velocity potential on the movable boundary. For this system, we prove the unique solvability on a small time interval in function classes similar to those introduced in [2] in the proof of the solvability of the Birkhoff equation. The main idea of our study is to introduce a scale of Banach spaces of special form and use the local existence and uniqueness theorem proved in [3] for an abstract differential equation in a scale of Banach spaces.

1. STATEMENT OF THE PROBLEM

Suppose that the interface at time $t$ is a smooth closed line $L_t$ dividing the plane into domains $D^+ = D_1$ (the external domain) and $D^- = D_2$ (the internal domain) filled by fluids with viscosities $\mu_1$ and $\mu_2$ and densities $\varrho_1$ and $\varrho_2$, respectively. The velocity of the filtering fluid is sought in the form $\vec{W} = \vec{W}_0 + \vec{W}^*$, where $\vec{W}_0$ is a given stationary velocity field exciting the flow,

$$\vec{W}_0(\vec{r}) = \sum_{n=1}^{N} Q_n V_q(\vec{r} - \vec{r}_n) + \vec{W}^*_0,$$

$\vec{r} = (x, y)$ is the position vector of the point on the plane at which the velocity is to be found, $V_q(\vec{r}) = -(2\pi)^{-1}|\vec{r}|^{-2}\vec{r}$ is the velocity field induced by a unit sink, the $\vec{r}_n = (x_n, y_n)$ are the position vectors of the points at which sinks of given intensities $Q_n$ are placed, and $\vec{W}^*_0$ is a vector function.
infinitely differentiable on the entire plane and satisfying the condition $\bar{W}_0^* \to 0$ as $|\vec{r}| \to \infty$. We assume that none of the points $\vec{r}_n$ lies on the interface $L_t$.

The continuity equation $\operatorname{div} \bar{W} = 0$ and the Darcy law $\bar{W} = \operatorname{grad} \varphi$ should hold outside the points $\vec{r}_n$ and the line $L_t$, where $\varphi = \mu^{-1}(p \Pi - p)$, $p$ is the pressure, $\mu = \mu_1$, $g = \varrho_1$ in the domains $D_t$, and $\Pi$ is a given (continuous on the contour $L_t$) potential specifying the external mass forces. The conditions $p_+ = p_-$ and $W_0^+ = W_0^- = W_n$ should be satisfied on the line $L_n$, where $p^\pm$ and $W^\pm_0$ are the boundary values of the pressure and the normal velocity component of the fluid and $W_n$ is the normal velocity of the motion of points of discontinuity.

Let $\vec{r}(s, t)$, $s \in R$, be a vector function that is $2\pi$-periodic in $s$ and parametrically defines the contour $L_t$ at time $t$. (Here $s$ is not necessarily a natural parameter.) Suppose that $\bar{W}_0 = \operatorname{grad} \varphi_0$ everywhere outside the points $\vec{r}_n$ and the contour $L_t$ and $\bar{W}^* = \operatorname{grad} \varphi^*$; moreover, the function $\bar{W}^*$ is continuously differentiable everywhere outside the contour $L_t$ (and also at the points $\vec{r}_n$).

Then the potential $\varphi^*$ should satisfy the relations

$$\Delta \varphi^* = 0 \quad \text{in} \quad D^+ \quad \text{and} \quad D^-, \quad (\partial \varphi/\partial n)^+ = (\partial \varphi/\partial n)^- \quad \text{on the contour} \quad L_t,$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator. For each time $t$, we seek the potential $\varphi^*$ under the additional condition $\varphi^* \to 0$ as $|\vec{r}| \to \infty$ in the form of the double layer potential

$$\varphi^*(\vec{r}_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) \frac{(\vec{r}_0 - \vec{r}) \vec{n}}{|\vec{r}_0 - \vec{r}|^2} \, ds, \quad \vec{r} = \vec{r}(s, t),$$

where $\vec{n}$ is the outward normal to the contour $L_t$ at the point $\vec{r} = \vec{r}(s, t)$. The condition $p^+ = p^-$ on the contour $L_t$ at time $t$ is equivalent to the condition $(1 - \lambda)\varphi^* - (1 + \lambda)\varphi^* = f$, where $f(s) = 2\lambda \varphi_0(\bar{r}) + 2\lambda_m \Pi(\bar{r})$, $\lambda = (\mu_2 - \mu_1)/(\mu_2 + \mu_1)$, $\lambda_m = (\varrho_2 - \varrho_1)/(\mu_2 + \mu_1)$, and $\bar{r} = \vec{r}(s, t)$. By using the formulas in [4, p. 320] for the jump of a double layer potential, we obtain the equation

$$\sigma^{-1}(s_0)g(s_0) - 2\lambda \int_{-\pi}^{\pi} g(s) \frac{1}{2\pi} \frac{\varphi'(x_0 - x) - \varphi'(y_0 - y)}{(x - x_0)^2 + (y - y_0)^2} \, ds = f(s_0), \quad s_0 \in R, \quad (1)$$

where $\sigma = |\partial \vec{r}/\partial s|$, $x = x(s, t)$, $y = y(s, t)$, $x_0 = x(s_0, t)$, $y_0 = y(s_0, t)$, $x' = \partial x(s, t)/\partial s$, $y' = \partial y(s, t)/\partial s$ (the functions $g$, $f$, and $\sigma$ also depend on $t$), and $\lambda$ is a constant satisfying the condition $|\lambda| < 1$.

It was also shown in [1] that the evolution of the interface $L_t$ is described by the equation

$$\partial \vec{r}(s, t)/\partial t = \bar{U}(s, t), \quad \vec{r}(s, t)|_{t=0} = \vec{r}_0(s) \equiv (x(s_0), y(s_0)), \quad (2)$$

where $\bar{U}(s, t) = (\bar{W}^* + \bar{W}^-)/2$ and the boundary values of the vector $\bar{W}$ are taken at the point $\vec{r}(s, t)$.

Suppose that $x_0(s)$ and $y_0(s)$ are $2\pi$-periodic functions describing a convex contour $L_0$, the tangent vector $\vec{t} = \partial \vec{r}/\partial s \neq 0$ is defined for all $t$ at each point of the contour $L_t$, and the vectors $(\vec{n}, \vec{t})$ form a right-handed pair.

It follows from the formula for the gradient of a double layer potential [5, p. 101] (the Biot–Savard law) that

$$\bar{U}(s_0, t) = \bar{U}_0(s_0, t) + \int_{-\pi}^{\pi} \tilde{g}'(s) \bar{V}(\bar{r}(s_0, t) - \vec{r}(s, t)) \, ds, \quad s_0 \in R, \quad \tilde{g} = \sigma^{-1}g,$$

$$\bar{U}_0(s_0, t) = \bar{W}_0(\vec{r}(s_0, t)), \quad \bar{V}(\vec{r}) \equiv (V_1(\vec{r}), V_2(\vec{r})) = (-y, x)(2\pi(x^2 + y^2))^{-1},$$

$$\vec{r} = (x, y) \in R^2, \quad \vec{r} \neq 0. \quad (3)$$

We have thereby reduced the problem to Eq. (2) in which $\bar{U}(s_0, t)$ is the function given by (3) and the function $g$ should satisfy Eq. (1). More complete requirements on the smoothness of the initial condition and the solution will be stated below.