ORDINARY DIFFERENTIAL EQUATIONS

On the Relationship Between Solutions of Delay Differential Equations and Infinite-Dimensional Systems of Differential Equations

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Abstract—We study the limit properties of solutions for a class of systems of ordinary differential equations as the number of equations and a certain parameter grow unboundedly. We show that the sequence of functions formed by the last components of solutions of such systems has a repeated limit. The limit function is a solution of a delay differential equation. Estimates of the convergence rate are obtained.

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1. INTRODUCTION

The theory of delay differential equations was intensively developing in the second half of the 20th century. This was related to numerous applied problems whose analysis necessitated solving delay equations. Equations of this type arise in the description of processes whose rate is determined by their previous states. Such processes are often referred to as “delay processes” or “processes with aftereffect.” A reasonably complete introduction to the theory of delay differential equations can be found, e.g., in [1–3].

The present paper deals with yet another problem that leads to the study of delay differential equations. More precisely, we continue the analysis of relationships, established in [4], between solutions of a class of systems of ordinary differential equations of infinite dimension and solutions of equations of the form

\[
\frac{dy(t)}{dt} = f(y(t), qy(t - \tau)), \quad t > \tau. \tag{1.1}
\]

The Cauchy problem for the following system of ordinary differential equations was considered in [4]:

\[
\begin{align*}
\frac{dx}{dt} &= A_n(\tau, \theta) x + h(q\theta y_n), \\
\frac{dy_n}{dt} &= f(y_n, x_n), \\
x_1|_{t=0} = \cdots = x_n|_{t=0} = 0, \\
y_n|_{t=0} &= y_0,
\end{align*}
\tag{1.2}
\]

where \(A_n(\tau, \theta)\) is the bidiagonal \(n \times n\) matrix with main diagonal 

\[
(-\frac{n-1}{\tau}, \ldots, -\frac{n-1}{\tau}, -\theta)
\]

and with subdiagonal entries \((n - 1)/\tau, \ldots, h(\xi)\) is the column vector of the form

\[
h(\xi) = \text{col}(\xi, 0, \ldots, 0), \quad \xi = q\theta y_n, \quad \tau, q > 0
\]

are fixed parameters, and \(\theta > 0\); moreover, \(f(u, v)\) is a bounded function satisfying the Lipschitz condition with respect to both variables:

\[
\sup_{u, v \in \mathbb{R}} |f(u, v)| = F < \infty, \quad |f(u_1, v_1) - f(u_2, v_2)| \leq L_1|u_1 - u_2| + L_2|v_1 - v_2|. \tag{1.3}
\]
It was proved in [4] that some class of solutions of Eq. (1.1) can be represented as the repeated limit
\[
\lim_{\theta \to \infty} \lim_{n \to \infty} y_n(t, \theta) = y(t),
\]
where \( y_n(t, \theta) \) is the last component of the solution of the Cauchy problem (1.2). The limit relation (1.4) was proved for \( q \in (0, 1) \) on the interval \( (\tau, T_0] \), where \( \tau < T_0 < \min \left\{ \frac{1-q}{L_1}, \frac{1}{L_2} \right\} \).

In the present paper, we strengthen this result by proving the limit relation (1.4) for any parameter \( q > 0 \) on an arbitrary interval \( (\tau, T] \) and by estimating the convergence rate.

Note that relationships of this type between solutions of systems of ordinary differential equations of infinite size and solutions of delay differential equations were obtained in [5] when modeling unbranched multistage synthesis of a material [6]. The corresponding system of differential equations has the form
\[
\begin{align*}
\frac{dx_1}{dt} &= g(x_n) - \frac{n-1}{\tau}x_1, \\
\frac{dx_i}{dt} &= \frac{n-1}{\tau}(x_{i-1} - x_i), \quad i = 2, \ldots, n-1, \\
\frac{dx_n}{dt} &= \frac{n-1}{\tau}x_{n-1} - \theta x_n.
\end{align*}
\]

It was shown in [5] that if the number \( n \) of equations in system (1.5) tends to infinity and only the last components of the solution of the Cauchy problem with zero initial data \( x|_{t=0} = 0 \) are considered, then we obtain a uniformly convergent sequence
\[
x_n(t) \to y(t), \quad n \to \infty, \quad t \in [0, T];
\]
moreover, the limit function \( y(t) \) satisfies the identity
\[
\frac{dy(t)}{dt} = -\theta y(t) + g(y(t-\tau)), \quad t > \tau.
\]

Consequently, the function \( y(t) \) is a solution of the delay differential equation (1.1) with right-hand side
\[
f(u, v) = -\theta u + g(v).
\]
These results were generalized in [7] to a wide class of quasilinear differential equations.

2. SYSTEMS WITH INFINITELY MANY DIFFERENTIAL EQUATIONS

The last two components of the solution of problem (1.2) are solutions of the following system of integral equations:
\[
\begin{align*}
x_n(t, \theta) &= q\theta \int_0^t \psi_n(t-s, \theta)y_n(s, \theta) \, ds, \\
y_n(t, \theta) &= y_0 + \int_0^t f(y_n(s, \theta), x_n(s, \theta)) \, ds,
\end{align*}
\]
where
\[
\begin{align*}
\psi_n(t, \theta) &= \frac{e^{-\theta t}}{(1 - \theta \tau/(n-1))^{n-1}} S_n(t, \theta), \\
S_n(t, \theta) &= 1 - e^{-\frac{n-1}{\tau} \theta t} \sum_{k=0}^{n-2} \frac{(\omega t)^k}{k!}, \quad \omega = \frac{n-1}{\tau} - \theta.
\end{align*}
\]