PARTIAL DIFFERENTIAL EQUATIONS

Blow-Up of Solutions of Some Nonlinear Wave Equations

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Abstract—We analyze the problem of blow-up of global solutions of a semilinear wave equation with a potential and with a possible degeneration at infinity for nonnegative initial data with compact support. By using the nonlinear capacity method, we prove the theorem on the nonexistence of such a solution for a subcritical and the critical nonlinearity exponent.

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1. INTRODUCTION

In the present paper, we consider the blow-up of solutions of the Cauchy problem for the following semilinear wave equation with nonnegative compactly supported initial data:

\[ u_{tt} - \Delta u + V(x)u = W(x)|u|^p \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+, \]

\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \mathbb{R}^N. \]  

(S1.1)

Such problems arise in nonlinear scattering theory [1]. Our aim is to obtain sufficient conditions for the blow-up of solutions of problem (1.1) with arbitrary initial data in the considered class in finite time.

To state these conditions, one should introduce the critical exponent \( p_c(N) \) of problem (1.1) defined as the positive root of the quadratic equation

\[(N - 1)p^2 - (N + 1 - 2\alpha)p - 2 = 0. \]  

(S1.2)

In accordance with the so-called Strauss conjecture [1], if \( \alpha = 0 \), \( V \equiv 0 \), and \( 1 < p \leq p_c \), then all solutions of problem (1.1) with nonnegative compactly supported initial data blow up in finite time. On the other hand, if \( p > p_c \) and \( V \equiv 0 \), then for sufficiently small regular data, there exist global solutions.

Results on the blow-up of solutions for \( 1 < p < p_c \) and their existence for \( p > p_c \) (throughout the following, we assume that \( V \equiv 0 \) and \( \alpha = 0 \) unless the contrary is stipulated) were essentially proved in [2] for \( N = 3 \) and in [3, 4] for \( N = 2 \). It was shown in [5] that there exist global solutions for small initial data for \( p > p_c(N) \) and \( N \geq 4 \). The corresponding results on the blow-up of solutions for \( 1 < p < p_c(N) \) and \( N \geq 4 \) were obtained in [6]. The critical case with zero potential was considered in [7] for \( N = 2, 3 \) and in [8] for \( N \geq 4 \). Results on the blow-up of solutions for \( N \geq 3, 1 < p < p_c(N) \), and for potentials \( V \) satisfying natural decay conditions [see (2.3) below] were obtained in [9]. We generalize this result to the critical case in which \( p = p_c(N) \) with \( V \neq 0 \) and \( \alpha \neq 0 \). Note also important theorems on the nonexistence of global solutions for systems of equations of the form (1.1) obtained in [10, 11].

The below-represented proof is based on the contradiction between the lower and upper a priori estimates for solutions. Upper estimates are obtained by the Mitidieri–Pokhozhaev nonlinear capacity method based on the use of regularized characteristic functions as test functions (see [12] and the bibliography therein). To obtain lower estimates, we use the properties of the Radon transform of solutions of (1.1) by analogy with [8, 9] but with some modifications that permit one to cover the critical case with nonzero potential.
2. STATEMENT OF THE PROBLEM

Consider the Cauchy problem for the semilinear wave equation (1.1), where
\[ W(x) \geq C(1 + |x|)^{-\alpha} \quad (x \in \mathbb{R}^N) \]  
(2.1)
with some \( C \geq 0 \) and \( \alpha \in \mathbb{R} \) and the initial data satisfy the conditions
\[ (u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N), \quad u_0(x) = u_1(x) = 0 \quad \text{if} \quad |x| > R > 0 \]  
(2.2)
for \( 1 < p \leq p_c(N, \alpha) \); moreover, \( N > (2p - \alpha)/(p - 1) \). Here \( p_c(N, \alpha) \) is the positive root of the quadratic equation (1.2). One can readily see that \( p_c(N, \alpha) > 1 \) for \( \alpha < 2 \).

We assume that the potential \( V : \mathbb{R}^N \to \mathbb{R} \) is a Hölder continuous function satisfying the estimates
\[ 0 \leq V(x) \leq \frac{C}{1 + |x|^{2+\delta}} \quad (x \in \mathbb{R}^N) \]  
(2.3)
with some constants \( C, \delta > 0 \).

Remark 2.1. Since \( N > (2p - \alpha)/(p - 1) \), without loss of generality, one can assume that \( 0 < \delta < (N(p - 1) + \alpha)/p - 2 \).

Remark 2.2. If inequalities (2.3) hold, then there exist functions \( \varphi_0, \varphi_1 \in C^2(\mathbb{R}^N) \) such that
\[ \Delta \varphi_0 - V \varphi_0 = 0 \quad \text{in} \quad \mathbb{R}^N, \quad c_0 < \varphi_0(x) \leq c_0^{-1}, \]
\[ \Delta \varphi_1 - V \varphi_1 = \varphi_1 \quad \text{in} \quad \mathbb{R}^N, \quad 0 < \varphi_1(x) \leq c_1(1 + |x|)^{-(N-1)}/2e^{|x|} \]
with some constants \( c_0, c_1 > 0 \) (see Lemma 3.1 in [8]). In particular, for \( V \equiv 0 \), one can set
\[ \varphi_0(x) = 1, \quad \varphi_1(x) = \int_{S^N_{x-1}} e^{x \cdot \omega} d\omega. \]

Let us state the main result of the present paper.

Theorem 2.1. Let the functions \( u_0 \) and \( u_1 \) be nonnegative almost everywhere in \( \mathbb{R}^N \), and let \( u_0 + u_1 \neq 0 \). In addition, let problem (1.1) have a solution \( (u, u_t) \in C([0, T_1], H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)) \) such that
\[ \text{supp}(u, u_t) \subset \{(x, t) : |x| \leq t + R\}. \]
If \( \alpha < \min\{2, (N - 1)(1 - p/2)\} \) and \( 1 < p \leq \min\{p_c(N, \alpha), 2\} \), then \( T_1 < \infty \).

3. AUXILIARY RESULTS

To prove this theorem, we need a number of a priori estimates. The below-represented Lemmas 3.1 and 3.2 are immediate generalizations of Yordanov and Zhang’s results [8, 9] to the case in which \( \alpha \neq 0 \), and Lemma 3.3 can be proved with the use of an appropriate modification of the Mitidieri–Pokhozhaev nonlinear capacity method (see [12]).

Lemma 3.1. Let the locally Hölder continuous potential \( V \) satisfy the estimate (2.3), let the functions \( u_0 \) and \( u_1 \) be nonnegative almost everywhere in \( \mathbb{R}^N \), and let \( u_0 + u_1 \neq 0 \). In addition, let \( u \) be a solution of the problem
\[ \Delta u - V(x)u - u_{tt} + W(x)|u|^p \leq 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \mathbb{R}^N, \]  
(3.1)
where \( W \) is a function satisfying condition (2.1) and the inclusion \( u(\cdot, t) \in H^1_0(B_{t+R}(0)) \) holds for all \( t \). Then the function
\[ F_0(t) := \int_{\mathbb{R}^N} u(x, t)\varphi_0(x) \, dx, \]  
(3.2)