The $R_{\nu}$-Generalized Solution of a Boundary Value Problem with a Singularity Belongs to the Space $W^{k+2}_{2,\nu+\beta/2+k+1}(\Omega, \delta)$

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Abstract—In the present paper, for a boundary value problem with noncoordinated degeneration of the data and a singularity in the solution, we show that the $R_{\nu}$-generalized solution belongs to the weighted space $W^{k+2}_{2,\nu+\beta/2+k+1}(\Omega, \delta)$ $(k > 0)$.

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INTRODUCTION

Natural restrictive conditions were obtained in [1, 2] for a boundary value problem with noncoordinated degeneration of the data and with a strong singularity in the solution. Hence, these conditions permit one to single out a unique solution from the pencil of $R_{\nu}$-generalized solutions and prove the coercivity inequality for it in a weighted Sobolev space. In the present paper, we show that the $R_{\nu}$-generalized solution belongs to the weighted space $W^{k+2}_{2,\nu+\beta/2+k+1}(\Omega, \delta)$ $(k > 0)$.

1. MAIN DEFINITIONS AND AUXILIARY ASSERTIONS

By $R^2$ we denote the two-dimensional Euclidean space; $x = (x_1, x_2)$ is an arbitrary point in it, $|x|^2 = x_1^2 + x_2^2$, and $dx = dx_1 dx_2$. Let $\Omega \subset R^2$ be a convex bounded domain with piecewise smooth boundary $\partial \Omega$, and let $\overline{\Omega}$ be the closure of $\Omega$, so that $\overline{\Omega} = \Omega \cup \partial \Omega$. By $\bigcup_{i=1}^n \tau_i$ we denote the set of singularity points $\theta_i$, $i = 1, \ldots, n$, of the boundary $\partial \Omega$, which includes the points of intersection of its smooth parts. By $O_i^\delta$ we denote the disk of radius $\delta$ with center $\tau_i$, $i = 1, \ldots, n$; i.e., $O_i^\delta = \{x : |x - \tau_i| \leq \delta\}$; we assume that $O_i^\delta \cap O_j^\delta = \emptyset$, $i \neq j$. Let $\Omega' = \bigcup_{i=1}^n \Omega_i$, where $\Omega_i = \Omega \cap O_i^\delta$, $i = 1, \ldots, n$.

We introduce the weight function $\varrho(x)$ that coincides in the neighborhood $\Omega_i$ of each point $\tau_i$, $i = 1, \ldots, n$, with the distance to it (i.e., $\varrho(x_1, x_2) = [(x_1 - x_1^i)^2 + (x_2 - x_2^i)^2]^{1/2}$, $(x_1^i, x_2^i) = \tau_i$) and is equal to $\delta$ for $x \in \overline{\Omega} \setminus \Omega'$. In addition, let the derivatives of the function $\varrho(x)$ satisfy the inequality $|\partial^k \varrho(x)/\partial x_1^{k_1} \partial x_2^{k_2}| \leq \sigma q^{m-k}(x)$.

By $W^l_{2,\alpha}(\Omega, \delta)$, $l \geq 1$, we denote the set of functions satisfying the following conditions:
(a) $|D^k u(x)| \leq c_1 \gamma^k k! / \varrho^{\alpha+k}(x)$, $x \in \Omega'$, where $k = 0, \ldots, l$ and $c_1, \gamma \geq 1$ are constants independent of $k$;
(b) $\|u(x)\|_{L^2(\Omega, \delta)} \geq c_2$, $c_2 = \text{const}$, with the squared norm

$$\|u(x)\|_{W^l_{2,\alpha-1}(\Omega, \delta)}^2 = \sum_{|\lambda| \leq l} ||\varrho^{\alpha-l-1} D^\lambda u(x)||_{L^2(\Omega)}^2,$$

where $D^\lambda u(x) = \partial^{|\lambda|} u / \partial x_1^{\lambda_1} \partial x_2^{\lambda_2}$, $|\lambda| = \lambda_1 + \lambda_2$, $\lambda_i \geq 0$, $i = 1, 2$, and $\alpha$ is a nonnegative real number.
By $W^{k}_{2,h} (\Omega, \delta) \subset W^{k}_{2,h} (\Omega, \delta)$ we denote the subset of functions that are zero almost everywhere on $\partial \Omega$, and by $L^{k}_{\infty,-\alpha} (\Omega, c_3)$ and $H^{k}_{\infty,-\alpha} (\Omega, c_4)$ we denote the sets of functions with the norms

$$\| u(x) \|_{L^{k}_{\infty,-\alpha}(\Omega, c_3)} = \text{vrai max}_{x \in \Omega} |g^{-\alpha}(x) u(x)| \leq c_3,$$

$$\| u(x) \|_{H^{k}_{\infty,-\alpha}(\Omega, c_4)} = \max_{|\lambda| \leq k} \text{vrai max}_{x \in \Omega} \left| g^{-\alpha+|\lambda|}(x) \frac{\partial^{(\lambda)} u(x)}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2}} \right| \leq c_4,$$

respectively.

For a real number $\alpha$, we set

$$L^{2,\alpha}(\Omega, \delta) = \left\{ u(r, \theta) \mid \| u(r, \theta) \|^2_{L^{2,\alpha}(\Omega, \delta)} = \int_{\Omega} r^{2\alpha} u^2(r, \theta) \, ds \right\},$$

where $ds = r \, dr \, d\theta$ and $(r, \theta)$ are local polar coordinates.

By $W^{l}_{2,\alpha}(\Omega, \delta)$, $l \geq 1$, we denote the set of functions satisfying the following conditions:

(a') $\| D^{\lambda} u(r, \theta) \| \leq \tilde{c}_1 \tilde{\gamma} |\lambda|! / r^{\alpha+\lambda_1}$, $(r, \theta) \in \Omega'$;

(b') $\| u(r, \theta) \|^2_{L^{2,\alpha}(\Omega, \Omega')} \geq \tilde{c}_2$, $\tilde{c}_2 = \text{const}$, with the squared norm

$$\| u(r, \theta) \|^2_{W^{l}_{2,\alpha+l-1}(\Omega, \delta)} = \sum_{|\lambda| \leq l} \| r^{\alpha+l-1-\lambda_2} |D^{\lambda} u(r, \theta)| \|^2_{L^{2,\alpha}(\Omega, \delta)},$$

where $D^{\lambda} u = \partial^{(\lambda)} u / \partial r^{\lambda_1} \partial \theta^{\lambda_2}$, $|\lambda| = 0, \ldots, l$, and $\tilde{c}_1, \tilde{\gamma} \geq 1$ are constants independent of $\lambda_1$.

By $L^{k}_{\infty,-\alpha}(\Omega, \tilde{c}_3)$ and $H^{k}_{\infty,-\alpha}(\Omega, \tilde{c}_4)$, we denote the sets of functions with the norms

$$\| u(r, \theta) \|_{L^{k}_{\infty,-\alpha}(\Omega, \tilde{c}_3)} = \text{vrai max}_{(r, \theta) \in \delta} |r^{-\alpha} u(r, \theta)| \leq \tilde{c}_3,$$

$$\| u(r, \theta) \|^2_{H^{k}_{\infty,-\alpha}(\Omega, \tilde{c}_4)} = \max_{|\lambda| \leq k} \text{vrai max}_{(r, \theta) \in \delta} \left| r^{\lambda_1-\alpha} \frac{\partial^{(\lambda)} u(r, \theta)}{\partial r^{\lambda_1} \partial \theta^{\lambda_2}} \right| \leq \tilde{c}_4,$$

respectively.

**Lemma 1.1.** For any function $u(x) \in W^{k+1}_{2,h} (\Omega, \delta)$, there exists a parameter $\alpha^*$ such that

$$\| u(x) \|_{W^{k+1}_{2,\alpha^*,-\alpha}(\Omega, \delta)} \leq c_5 \| u(x) \|_{W^{k}_{2,\alpha}(\Omega, \delta)},$$

where $0 < c_5 < 1$.

The proof of Lemma 1.1 is similar to that of Lemma 1.1 in [2, p. 534].

2. STATEMENT OF THE PROBLEM

In the domain $\Omega$, consider the equation

$$- \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( a_{ii}(x) \frac{\partial u(x)}{\partial x_i} \right) + a(x) u(x) = f(x), \quad x \in \Omega,$$

with the boundary condition

$$u(x) = 0, \quad x \in \partial \Omega.$$

**Definition 2.1.** The boundary value problem (2.1), (2.2) is called the Dirichlet problem with noncoordinated degeneration of the data if the coefficients of the equation satisfy the conditions

$$a_{ii}(x) \in H^{k+1}_{\infty,-\beta}(\Omega, c_6), \quad a(x) \in H^{k}_{\infty,-\beta}(\Omega, c_7),$$

$$\sum_{l=1}^{2} a_{il}(x) \xi_l^2 \geq c_8 \vartheta^{\delta}(x) \sum_{l=1}^{2} \xi_l^2, \quad a(x) \geq c_9 \vartheta^{\delta}(x)$$

where $\vartheta(x)$ and $\delta(x)$ are functions depending on $x$.

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