Optimization in the Class $W^2_2$ of the Boundary Control by Displacements at One Endpoint of a String with the Other Endpoint Being Fixed

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Abstract—We consider the problem on the optimal boundary control of string vibrations by a displacement at one endpoint of the string with the other endpoint being fixed. The problem is studied in the space $\hat{W}^2_2$ and then, more generally, in the space $\hat{W}^p_2$ for $p \geq 1$.

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STATEMENT OF THE PROBLEM

The present paper, which is a natural continuation of [1], deals with elastic string vibrations described by the one-dimensional wave equation

$$u_{tt}(x, t) - u_{xx}(x, t) = 0$$

in the rectangle $Q_T = [0 < x < l] \times [0 < t < T]$, where $l$ is the string length and the vibrations are observed on a time interval $T$ that is a multiple of the double length of the string, $T = 2l(n + 1)$, $n = 1, 2, \ldots$ The right endpoint of the string is fixed,

$$u(l, t) = 0,$$

and the control by displacements is performed at the left endpoint,

$$u(0, t) = \mu(t).$$

This control brings the string from an arbitrarily given initial state

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 \leq x \leq l,$$

to an arbitrary terminal state

$$u(x, T) = \hat{\varphi}(x), \quad u_t(x, T) = \hat{\psi}(x), \quad 0 \leq x \leq l,$$

where the initial and terminal functions satisfy the fixing conditions $\varphi(l) = \hat{\varphi}(l) = 0$ and $\psi(l) = \hat{\psi}(l) = 0$, following from (2).

The subsequent investigation will be carried out in terms of a generalized solution of the wave equation. It was suggested in [2, 3] and other papers to treat a generalized solution as a function $u(x, t)$ in the space $\hat{W}^1_2[Q_T]$ (or $\hat{W}^p_2[Q_T]$, $p \geq 1$, in a more general setting). In the present paper, we construct a generalized solution in the space $\hat{W}^2_2[Q_T]$ with increased smoothness. Boundary control problems were already considered in this class with $p = 2$ for critical and subcritical time intervals, e.g., in [4].
Definition 1. A function \( u(x, t) \) belongs to the space \( \hat{W}^2_p[Q_T] \) if it and its partial derivatives are continuous in the closed rectangle \( Q_T \) and if \( u(x, t) \) has all generalized second partial derivatives each of which belongs to the class \( L_p(0, l) \) for each \( t \in [0, T] \) and to the class \( L_p(0, T) \) for each \( x \in [0, l] \).

The requirement that the generalized solution \( u(x, t) \) belongs to the class \( \hat{W}^2_p(Q_T) \) permits precisely stating the smoothness conditions for the functions occurring in the control and in the initial and terminal conditions:

\[
\varphi(x), \hat{\varphi}(x) \in W^2_p[0, l], \quad \psi(x), \hat{\psi}(x) \in W^1_p[0, l], \quad \mu(t) \in W^2_p[0, T].
\]

(6)

It is known that, for sufficiently large time intervals \( T > 2l \), there exist infinitely many solutions \( \mu(t) \) of the above-posed boundary control problem; therefore, we consider the problem of finding a so-called optimal boundary control. To this end, of all \( \mu \in W^2_p[0, T] \) we choose a control minimizing the integral

\[
\int_0^T |\mu''(t)|^p \, dt.
\]

(7)

MIXED PROBLEM

Consider the corresponding mixed problem for Eq. (1) with the boundary conditions (2) and (3) and the initial conditions (4). In addition, we require that the control function \( \mu(t) \) satisfies the matching conditions \( \mu(0) = \varphi(0), \mu'(0) = \psi(0), \mu(T) = \hat{\varphi}(0) \), and \( \mu'(T) = \hat{\psi}(0) \). The mixed problem (1)–(4) is also considered in terms of a generalized solution in the class \( \hat{W}^2_p(Q_T) \).

Definition 2. A generalized solution of the mixed initial-boundary value problem (1)–(4) in the class \( \hat{W}^2_p(Q_T) \) is a function \( u(x, t) \in \hat{W}^2_p(Q_T) \) satisfying the integral identity

\[
\int_0^T \int_0^l u(x, t)\left[ \Phi_{tt}(x, t) - \Phi_{xx}(x, t) \right] \, dx \, dt + \int_0^l \varphi(x)\Phi_t(x, 0) \, dx - \int_0^l \psi(x)\Phi(x, 0) \, dx - \int_0^T \mu(t)\Phi_x(0, t) \, dt = 0
\]

(8)

for every function \( \Phi(x, t) \) in the class \( C^2(\hat{Q}_T) \) satisfying the conditions

\[
\Phi(x, T) \equiv 0, \quad \Phi_t(x, T) \equiv 0 \quad \text{for any} \quad 0 \leq x \leq l,
\]

\[
\Phi(0, t) = 0, \quad \Phi(l, t) \equiv 0 \quad \text{for any} \quad 0 \leq t \leq T.
\]

(9)

Theorem 1. There exists a unique generalized solution of the mixed problem (1)–(4) in the space \( \hat{W}^2_p \) for \( p \geq 1 \).

Proof. First, let us construct the solution of the mixed initial-boundary value problem in closed analytic form. To this end, we continue the initial and terminal functions \( \varphi(x), \psi(x), \hat{\varphi}(x), \) and \( \hat{\psi}(x) \) as odd functions around the point \( x = l \) from the interval \([0, l]\) to the interval \([l, 2l]\) and consider the function \( \tilde{u}(x, t) \) given by the relation

\[
\tilde{u}(x, t) = \begin{cases} 
\frac{1}{2} \left[ \varphi(x + t) + \varphi(x - t) + \int_{x-t}^{x+t} \psi(\tau) \, d\tau \right] & \text{in } \Delta_1 \\
\frac{1}{2} \left[ \varphi(x + t) + \varphi(0) + \int_0^{x+t} \psi(\tau) \, d\tau + (x-t)(\varphi'(0) - \psi(0)) \right] & \text{in } \Delta_2 \\
(\varphi'(0) - \psi(0))(x-l) & \text{in } \Delta_3,
\end{cases}
\]

(10)