Boundary Control Problem
for the First Boundary Value Problem
for a Second-Order System of Hyperbolic Type

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Abstract—We consider a boundary control problem for a system of second-order hyperbolic equations without the mixed derivative. The boundary functions are constructed. We state a theorem that gives existence conditions for boundary controls.

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1. INTRODUCTION

In the recent years, control problems for elastic vibrations were considered in [1–9] and other papers, in which the solutions of problems of controllability of elastic vibrations by boundary controls for various types of boundary conditions were suggested.

In the course of its development, control theory has undergone several modifications: the passage from classical solutions to generalized solutions of the equation of state of the system [1–3, 9, 10], the investigation of more complicated equations describing the state of the system [5, 7–8, 11], and the introduction of more complicated boundary conditions [12, 13].

In the present paper, the results obtained by V.A. Il’in for the boundary control problem for the homogeneous wave equation are transferred to the case of plants described by the second-order hyperbolic system

\[ w_{tt} - Aw_{xx} = 0, \]

(1)

where \( A \) is a constant \( m \times m \) matrix with positive real eigenvalues, \( w(x, t) = \text{col}(w_1, w_2, \ldots, w_m) \) is a vector function, and \( w_i \in C^2(Q), i = 1, \ldots, m \).

2. GENERAL SOLUTION OF THE MATRIX WAVE EQUATION

Consider system (1) in the domain \( Q_{l,T} = [0, l] \times [0, T] \). Suppose that the characteristic equation of the matrix \( A \) has a positive root \( \lambda \) of multiplicity \( m \). By [15, pp. 141–145], there exists a matrix \( S \) reducing the matrix \( A \) to the form \( \Lambda = S^{-1}AS \), where \( \Lambda \) is a Jordan block similar to the matrix \( A \) (in the case of multiple eigenvalues) or a diagonal matrix whose principal diagonal consists of distinct eigenvalues. The transformation matrix \( S \) reducing the matrix \( A \) to block diagonal form consists of vectors of all series corresponding to all roots of the characteristic equation.

By the change of variables \( u = S^{-1}w \), we reduce system (1) to the form

\[ u_{tt} - \Lambda u_{xx} = 0 \]

(2)

in the domain \( Q_{l,T} \), where \( u(x, t) = \text{col}(u_1, u_2, \ldots, u_m) \) is a vector function.

If \( \Lambda \) is a Jordan block, then system (2) is a recursion system of inhomogeneous wave equations

\[ \frac{\partial^2}{\partial t^2} u_i - \lambda^2 \frac{\partial^2}{\partial x^2} u_i = \frac{\partial^2}{\partial x^2} u_{i-1}, \quad i = 1, \ldots, m, \quad u_0(x, t) = 0. \]

(3)
The general solution of the $i$th equation in system (3) for $i \geq 2$ has the form [16]

$$u_i = \sum_{k=1}^{i} \frac{\delta(i-k)u_j^0(x, t)}{(i-k)!} + \sum_{k=1}^{i-1} \frac{(-1)^k C_{2k-1}^k}{(2\lambda)^{2k}} \sum_{m=1}^{i-k} \frac{\delta(i-k-m)(u_m^0(x, t) - u_m^0(x, -t))}{(i-k-m)!},$$

(4)

where $\delta \equiv \frac{1}{2\lambda} \frac{\partial}{\partial \lambda}$, $\delta^0 \equiv 1$, and $u_j^0$ is the general solution of the corresponding homogeneous wave equation with index $j$, which is given by the relation

$$u_j^0(x, t) = f_j(x + \lambda t) + g_j(x - \lambda t);$$

(5)

here $f_j$ and $g_j$ are arbitrary functions of the class $C^l$, $l = 2 + m - j$, and $m$ is the order of the matrix $A$.

If the matrix $A$ has $m$ distinct eigenvalues, then the matrix equation (2) is equivalent to a system of homogeneous wave equations, whose general solution is given by formula (5) with parameter $\lambda = \lambda_j$, $j = 1, \ldots, m$.

3. BOUNDARY CONTROL PROBLEM

For system (1) in the domain $Q_{l, T}$ for $T < l/\lambda$, we pose the initial conditions

$$w(x, 0) = \varphi(x), \quad w_t(x, 0) = \psi(x), \quad 0 \leq x \leq l,$$

(6)

the terminal conditions

$$w(x, T) = \hat{\varphi}(x), \quad w_t(x, T) = \hat{\psi}(x), \quad 0 \leq x \leq l,$$

(7)

and the boundary conditions of the first kind

$$w(0, t) = \mu(t), \quad w(l, t) = \nu(t), \quad 0 \leq t \leq T,$$

(8)

where $\varphi(x)$, $\psi(x)$, $\hat{\varphi}(x)$, $\hat{\psi}(x)$, $\mu(t)$, and $\nu(t)$ are $m$-vector functions.

**Problem 1.** Find vector functions $\mu(t)$ and $\nu(t)$ such that the classical solution $w(x, t)$ of the first boundary value problem with given initial conditions $[\varphi(x), \psi(x)]$ satisfies the terminal conditions with given vector functions $[\hat{\varphi}(x), \hat{\psi}(x)]$ at time $t = T$.

Consider the boundary control problem as a problem of bringing the system from a given state $[\varphi(x), \psi(x)]$ into a state $[\hat{\varphi}(x), \hat{\psi}(x)]$ in a given time $T$.

To solve the boundary control problem, we study its special cases: the problem on the complete damping of the system and the problem of bringing the originally quiescent system into a given state. We find the solution of the boundary control problem as the sum of solutions of these special problems.

4. PROBLEM ON THE COMPLETE DAMPING OF THE SYSTEM

**Problem 2.** Find vector functions $\mu(t)$ and $\nu(t)$ such that the classical solution $w(x, t)$ of the first boundary value problem in the domain $Q_{l, T}$, $T < l/\lambda$, with given initial conditions $[\varphi(x), \psi(x)]$ satisfies the zero terminal conditions $w(x, T) = 0$ and $w_t(x, T) = 0$ at time $t = T$.

By using the change of variables $u = S^{-1}w$, we reduce system (1) to the system of $m$ inhomogeneous wave equations with the special right-hand side (3). The initial conditions (6), the terminal conditions (7), and the boundary conditions (8) acquire the form

$$u(x, 0) = S\varphi(x), \quad u_t(x, 0) = S\psi(x), \quad 0 \leq x \leq l,$$

(9)

$$u(x, T) = S\hat{\varphi}(x), \quad u_t(x, T) = S\hat{\psi}(x), \quad 0 \leq x \leq l,$$

(10)

$$u(0, t) = S\mu(t), \quad u(l, t) = S\nu(t), \quad 0 \leq t \leq T,$$

(11)

respectively.