ORDINARY DIFFERENTIAL EQUATIONS

On the Construction of First-Order Polynomial Differential Equations Equivalent to a Given Equation in the Sense of Having the Same Reflective Function

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Abstract—We consider a technique for constructing first-order differential equations with polynomial right-hand sides and with Mironenko reflective function coinciding with that of a given polynomial equation. We study relations between equations constructed by this technique.

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We apply the reflective function theory [1] to the analysis of the equation

\[
\dot{x} = a_0(t) + a_1(t)x + \cdots + a_n(t)x^n, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}.
\]

(1)

Let us give information we need from the reflective function theory [2, pp. 62–69; 3, pp. 11–16]. For a differential system

\[
\dot{x} = X(t, x), \quad x = (x_1, x_2, \ldots, x_l)^T \in \mathbb{R}^l, \quad t \in \mathbb{R},
\]

(2)

with general solution \(\varphi(t; t_0, x_0)\), the reflective function is defined by the formula

\[
F(t, x) := \varphi(-t; t, x).
\]

If \(F(t, x)\) is the reflective function of the differential system (2) and \(x(t)\) is an arbitrary solution of (2) defined for \(t = 0\), then \(F(t, x(t)) \equiv x(-t)\) for all \(t\) in any symmetric interval lying in the domain of \(x(t)\). Distinct differential systems may have the same reflective function. Systems whose reflective functions coincide in their common domain are said to be equivalent. Obviously, this relationship is an equivalence relation indeed. All systems in an equivalence class, and only these systems, can be represented in the form

\[
\dot{x} = -\frac{1}{2}F_x^{-1}F_t + F_x^{-1}R(t, x) - R(-t, F),
\]

where \(F(t, x)\) is the reflective function specifying the class, \(R(t, x)\) is an arbitrary continuous vector function, and \(F_t\) and \(F_x\) are the derivatives of \(F\) with respect to the corresponding variables. If system (2) is \(2\omega\)-periodic in \(t\), then \(F(-\omega, x)\) is its Poincaré map over the period \([-\omega, \omega]\). Equivalent systems have the same operators of translation [4, p. 11] along solutions on the symmetric time interval \([-\omega, \omega]\), and therefore, the initial data \(x(-\omega)\) of solutions of boundary value problems of the form \(\Phi(x(\omega), x(-\omega)) = 0\), where \(\Phi\) is an arbitrary function, coincide for such systems. A differentiable vector function \(F(t, x)\) is the reflective function of a system (2) whose right-hand side is continuously differentiable if and only if it satisfies the main relation

\[
F_t(t, x) + F_x(t, x)X(t, x) + X(-t, F(t, x)) \equiv 0
\]

(3)

and the initial condition \(F(0, x) \equiv x\).
It was shown in [5] (see also [2, pp. 170–180]) that if continuously differentiable vector functions
\[ \Delta_i(t, x) = (\Delta_{i1}(t, x), \Delta_{i2}(t, x), \ldots, \Delta_{ik}(t, x))^T, \quad i = 1, \ldots, m, \]
are solutions of the differential system
\[
\frac{\partial \Delta(t, x)}{\partial t} + \frac{\partial \Delta(t, x)}{\partial x} X(t, x) - \frac{\partial X(t, x)}{\partial x} \Delta(t, x) = 0, \tag{4}
\]
then all perturbed systems of the form
\[
\dot{x} = X(t, x) + \sum_{i=1}^{k} \alpha_i(t) \Delta_i(t, x), \tag{5}
\]
where the \( \alpha_i(t) \) are arbitrary continuous scalar odd functions, are equivalent to each other and to system (2). For \( k = \infty \), we assume that the series on the right-hand side in (5) is uniformly convergent.

Reflective functions were used for studying differential systems not only by V.I. Mironenko but also by L.A. Al’sevich, P.P. Veresovich, O.A. Kastritsa, E.V. Musafirov, Zhou Zhengxin, et al. A fairly complete bibliography on the topic can be found in [2].

Thus, for a given equation (1), we present a method for constructing other polynomial differential equations equivalent to (1). We assume in (4) and (5) that \( X(t, x) \) is the right-hand side of (1), and we use Eq. (4) when studying the differential equation (1); namely, we construct equations equivalent to (1) by finding solutions of (4). However, it is impossible to find all solutions in most cases. Therefore, we take only polynomial solutions of (4), that is, solutions of the form
\[
\Delta(t, x) = r_0(t) + r_1(t)x + \cdots + r_m(t)x^m, \tag{6}
\]
where the coefficients \( r_j(t), j = 1, \ldots, m, r_m(t) \neq 0 \), are assumed to be differentiable functions on \( \mathbb{R} \). This problem proves to be substantially simpler.

The number of functions \( \Delta_i(t, x) \) of polynomial form depends on the right-hand side of the considered equation (1). Obviously, if there is one solution \( \Delta(t, x) \), then there are infinitely many such solutions \( \Delta(t, x) \), because \( c \Delta(t, x) \) with arbitrary \( c = \text{const} \) is a solution of Eq. (4) as well. Therefore, to construct distinct perturbed equations (5), for such perturbations, one should use at least linearly independent solutions \( \Delta_i(t, x) \) of Eq. (4). Obviously, a perturbation constructed with the use of linearly dependent solutions \( \Delta_i(t, x) \) can also be constructed on the basis of fewer but linearly independent \( \Delta_i(t, x) \). However, the usual notion of linear independence is too broad to describe the entire set of systems of the form (5).

Indeed, suppose that the functions \( \Delta_1(t, x), \ldots, \Delta_k(t, x) \) are linearly independent, but some (at least one) of them, say, \( \Delta_k(t, x) \), can be represented as a linear combination of the remaining functions \( \Delta_i(t, x) \) with coefficients being even functions of \( t \); i.e.,
\[
\Delta_k(t, x) = m_k(t) \Delta_k(t, x) + \cdots + m_{k-1}(t) \Delta_{k-1}(t, x). \tag{7}
\]
One can readily see that every perturbation \( \alpha_i(t) \Delta_1(t, x) + \cdots + \alpha_k(t) \Delta_k(t, x) \) of Eq. (1), where the \( \alpha_i(t) \) range over the class of continuous odd functions, can be expressed only via \( \Delta_1(t, x), \ldots, \Delta_{k-1}(t, x) \), i.e., without the function \( \Delta_k(t, x) \) [obviously, with different odd functions \( \alpha_i(t) \), \( i = 1, \ldots, k - 1 \)]. To single out a minimal set of linearly independent solutions \( \Delta_i(t, x) \) of Eq. (4) sufficient for constructing an arbitrary perturbed equation (5) equivalent to Eq. (1), we introduce the following definition.

Let \( \mathcal{E} \) be the class of continuous even functions \( \mathbb{R} \rightarrow \mathbb{R} \). We say that functions \( \Delta_i(t, x), \quad i = 0, \ldots, k, \) are linearly independent over \( \mathcal{E} \) if there are no functions \( m_i(t) \in \mathcal{E}, \quad i = 0, \ldots, k, \) such that not all of them are identically zero and \( \sum_{i=0}^{k} m_i(t) \Delta_i(t, x) \equiv 0 \).

We shall choose linearly independent functions over \( \mathcal{E} \) from linearly independent solutions \( \Delta_i(t, x) \) of Eq. (4). In what follows, we show that equations of the form (1) with a polynomial of the same degree may have various numbers of linearly independent polynomial solutions \( \Delta(t, x) \).

**Lemma.** Suppose that the polynomial on the right-hand side in (1) has degree \( n \geq 2 \) and the set of zeros of the coefficient \( a_n(t) \) is nowhere dense in \( \mathbb{R} \). If Eq. (4) has a nonzero polynomial solution of the form (6), then \( m = n \).