Ordinary Differential Equations

Relaxation Self-Oscillations in Neuron Systems: III


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Abstract—The mathematical model considered here of a neuron system is a chain of an arbitrary number \( m \geq 2 \) of diffusion-coupled singularly perturbed nonlinear delay differential equations with Neumann-type conditions at the endpoints. We study the existence, asymptotic behavior, and stability of relaxation periodic solutions of this system.

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1. DESCRIPTION OF THE OBJECT OF STUDY

The mathematical model to be studied below has the form [1]

\[
\dot{u}_j = d(u_{j+1} - 2u_j + u_{j-1}) + \lambda[-1 + \alpha f(u_j(t-1)) - \beta g(u_j)]u_j, \quad j = 1, \ldots, m, \tag{1}
\]

where \( m \geq 2 \), \( u_{m+1} = u_m \), and \( u_0 = u_1 \). Here the \( u_j = u_j(t) > 0 \) are the neuron membrane potentials, \( \lambda > 0 \) is a large parameter, and \( \alpha, \beta > 0 \) are parameters of the order of unity satisfying the inequality

\[
\alpha > 1 + \beta. \tag{2}
\]

Following [2], we assume that the functions \( f(u) \) and \( g(u) \) belong to the class \( C^2(\mathbb{R}_+) \), \( \mathbb{R}_+ = \{u \in \mathbb{R} : u \geq 0\} \), and have the properties

\[
\begin{align*}
f(0) &= g(0) = 1, \quad 0 < \beta g(u) + 1 < \alpha \quad \forall u \in \mathbb{R}_+; \\
f(u), g(u), uf'(u), ug'(u), u^2f''(u), u^2g''(u) &= O(1/u) \quad \text{as} \quad u \to +\infty. \tag{3}
\end{align*}
\]

Under these conditions, system (1) admits the so-called homogeneous, or synchronous, cycle

\[
u_1 \equiv u_2 \equiv \cdots \equiv u_m = u_*(t, \lambda), \tag{4}
\]

where \( u_*(t, \lambda) \) is the stable periodic solution of the equation

\[
\dot{u} = \lambda[-1 + \alpha f(u(t-1)) - \beta g(u)]u \tag{5}
\]

with period

\[
T_*(\lambda) : \lim_{\lambda \to \infty} T_*(\lambda) = T_0, \quad T_0 = \alpha + 1 + (\beta + 1)/(\alpha - \beta - 1). \tag{6}
\]

Recall that the existence and uniqueness of the desired cycle of Eq. (5) were already established by the authors in [3].

In the following, we show that, first, the homogeneous cycle (4) of system (1) is exponentially orbitally stable for each \( d > 0 \) and for all \( \lambda \gg 1 \); second, in addition to the stable homogeneous cycle, this system has at least \( m \) exponentially orbitally stable inhomogeneous periodic motions for an appropriate choice of the parameters \( \alpha \) and \( \beta \) and appropriate reduction of the diffusion constant \( d \). By analogy with the spatially continuous case, these motions will be referred to as autowave modes.
2. BASIC THEOREM

Just as in the case \( m = 2 \), considered in [2], we carry out the subsequent analysis of system (1) in the new variables \( x, y_1, \ldots, y_{m-1} \), where

\[
u_1 = \exp\left(\frac{x}{\varepsilon}\right), \quad u_j = \exp\left(\frac{x}{\varepsilon} + \sum_{k=1}^{j-1} y_k\right), \quad j = 2, \ldots, m, \quad \varepsilon = \frac{1}{\lambda} \ll 1. \tag{7}\]

By substituting the expressions (7) into system (1), we arrive at the relaxation system

\[
\begin{align*}
\dot{x} &= \varepsilon d(\exp y_1 - 1) + F(x, x(t-1), \varepsilon), \\
\dot{y}_j &= d[\exp y_{j+1} + \exp(-y_j) - \exp y_j - \exp(-y_{j-1})] \\
&\quad + G_j(x, x(t-1), y_1, \ldots, y_j, y_1(t-1), \ldots, y_j(t-1), \varepsilon), \quad j = 1, \ldots, m-1,
\end{align*}
\tag{8}
\]

where \( y_0 = y_m = 0 \) and the functions \( F \) and \( G_j \) have the form

\[
\begin{align*}
F(x, u, \varepsilon) &= -1 + \alpha f\left(\exp\left(\frac{u}{\varepsilon}\right)\right) - \beta g\left(\exp\left(\frac{x}{\varepsilon}\right)\right), \\
G_j(x, u, y_1, v_1, \ldots, v_j, \varepsilon) &= \frac{\alpha}{\varepsilon}\left[ f\left(\exp\left(\frac{u}{\varepsilon} + \sum_{k=1}^{j} v_k\right)\right) - f\left(\exp\left(\frac{u}{\varepsilon} + \sum_{k=1}^{j-1} v_k\right)\right)\right] \\
&\quad + \frac{\beta}{\varepsilon}\left[ g\left(\exp\left(\frac{x}{\varepsilon} + \sum_{k=1}^{j-1} y_k\right)\right) - g\left(\exp\left(\frac{x}{\varepsilon} + \sum_{k=1}^{j} y_k\right)\right)\right], \quad j = 1, \ldots, m-1.
\end{align*}
\]

Take a constant \( \sigma_0 \) satisfying the conditions \( 0 < \sigma_0 < \min((\beta+1)/(\alpha-\beta-1), 1) \). On the interval \( -\sigma_0 \leq t \leq T_0 - \sigma_0 \), where \( T_0 \) is defined in (6), by \( y^m(t, z), \ldots, y^{m-1}_m(t, z), z = (z_1, \ldots, z_{m-1}) \in \mathbb{R}^{m-1} \), we denote the components of the solution of the impulse system

\[
\begin{align*}
\dot{y}_j &= d[\exp y_{j+1} + \exp(-y_j) - \exp y_j - \exp(-y_{j-1})], \\
y_j(+0) &= \frac{\alpha - 1}{\alpha - \beta - 1} y_j(-0), \quad y_j(1+0) = y_j(1-0) - \frac{\alpha}{\alpha - 1} y_j(+0), \\
y_j(0+) = (1 + \beta)y_j(0-), \quad y_j(0+0) = y_j(0+0) - \frac{\alpha}{1 + \beta} y_j(0+), \quad j = 1, \ldots, m-1, \\
y_0 = y_m = 0
\end{align*}
\tag{9}
\]

supplemented with the initial condition

\[
(y_1, \ldots, y_{m-1})|_{t=-\sigma_0} = (z_1, \ldots, z_{m-1}).
\tag{10}
\]

Next, consider the mapping

\[
z \rightarrow \Phi(z) \overset{\text{def}}{=} (y^0_1(t, z), \ldots, y^0_{m-1}(t, z))|_{t=T_0-\sigma_0}
\tag{11}
\]

of \( \mathbb{R}^{m-1} \) into \( \mathbb{R}^{m-1} \). The following assertion holds.

**Theorem 1.** To each fixed point \( z = z_* \), stable or dichotomous, of the mapping (11), there corresponds a relaxation cycle \( (x(t, \varepsilon), y_1(t, \varepsilon), \ldots, y_{m-1}(t, \varepsilon)) \), \( x(-\sigma_0, \varepsilon) \equiv -\sigma_0(\alpha-\beta-1) \), of period \( T(\varepsilon) \) of system (8) with the same stability properties for all sufficiently small \( \varepsilon > 0 \). In addition, one has the limit relations

\[
\begin{align*}
\lim_{\varepsilon \to 0} T(\varepsilon) &= T_0, \\
\lim_{\varepsilon \to 0} \max_{-\sigma_0 \leq t \leq T(\varepsilon) - \sigma_0} |x(t, \varepsilon) - x_0(t)| &= 0, \\
\lim_{\varepsilon \to 0} \max_{t \in \Sigma(\varepsilon)} |y_j(t, \varepsilon) - y^0_j(t, z_*)| &= 0, \quad j = 1, \ldots, m-1,
\end{align*}
\]