On the Number of Linear Particular Integrals of Algebraic Differential Equations

M. V. Dolov and E. V. Kruglov

Nizhni Novgorod State University, Nizhni Novgorod, Russia
e-mail: kruglov19@mail.ru

Received June 5, 2014

Abstract—We obtain a sharp bound for the number of distinct linear particular integrals of a polynomial vector field with coprime components the maximum of whose degrees is not less than 2.

DOI: 10.1134/S0012266115040138

Linear particular integrals were efficiently used for the solution of local and global problems in the theory of differential equations by L. Euler, C. Jacobi, F.G. Minding, N.N. Bautin, K.S. Sibirskii, N.I. Vulpe, M.N. Popa, A.S. Shube, T.A. Druzhkova, R.A. Lyubimova, and others. The statement of the problem in the present paper is related to [1].

Consider the system of differential equations

\[ \dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \]  

where \( P \) and \( Q \) are coprime polynomials whose coefficients, as well as the variables \( x \) and \( y \), are complex in the general case and \( \max(\deg P, \deg Q) = n \).

By definition, system (1) is algebraically integrable if all invariant sets of system (1) are algebraic. The following assertion was proved in [1].

Theorem 1. The maximum number \( S(n) \) of distinct irreducible (over the field of complex numbers) algebraic invariant curves of algebraically nonintegrable systems (1) satisfies the estimate \( S(n) \leq (n^2 + n + 2)/2 = p; \) moreover, \( S(2) = 4 \).

The sharpness of the estimate for \( S(3) \) was proved in [2, 3], where it was shown that the equalities \( S(2) = 4 \) and \( S(3) = 7 \) are attained in the class of systems (1) with linear particular integrals whose coefficients can be complex in the case of real systems. The problem on the sharpness of the estimate \( S(n) \leq p \) for \( n \geq 4 \) remains open.

In connection with the inequality \( S(n) \leq p \), one can ask whether the equality \( S(n) = p \) can be attained in the class of systems (1) with linear particular integrals.

The following assertion is the main result of the present paper.

Theorem 2. If \( n \geq 2 \), then system (1) with coprime polynomials \( P \) and \( Q \) has at most \( 3n - 1 \) distinct linear particular integrals, and this bound is sharp.

AUXILIARY ASSERTIONS

The following assertion was proved in [4].

Lemma. If the differential equation \( Q(x, y)dx - P(x, y)dy = 0 \), where \( P \) and \( Q \) are polynomials and \( \max(\deg P, \deg Q) = n \geq 2 \), admits a general integral

\[ F(x, y) - CG(x, y) = 0, \]  

where \( C \) is a constant.

563
where $C \in \mathbb{C}$ and $F(x, y)$ and $G(x, y)$ are linear functions, then the polynomials $P$ and $Q$ have a common divisor that is not identically constant.

In the general case, assuming that the coefficients of the polynomials $P$ and $Q$ and the variables $x$ and $y$ are complex, one can prove the following assertion, just as in [5, p. 40].

**Theorem 3** (V.I. Mironenko). If $r$ distinct algebraic curves $R_j(x, y) = 0$, $j = 1, \ldots, r$, deg($R_j$) = $m_j$, irreducible over the field of complex numbers are invariant for system (1) and

$$\sum_{j=1}^{r} m_j > \frac{1}{24} m(m + 1)(m + 2)(8 + 3(m + 3)(n - 1)),$$

where $m = \max_{1 \leq j \leq r} m_j$, then system (1) is algebraically integrable and the order of curves does not exceed $m$.

**Proof of Theorem 2.** The first part of Theorem 2 can be proved by analogy with [4]. Assume that there exist systems (1) with coprime polynomials $P$ and $Q$ with $3n$ distinct linear particular integrals. Inequality (3) holds for $r = 3n$ and $m_j = m = 1$. By Theorem 3, system (1) is algebraically integrable and has a general integral (2), where $F$ and $G$ are linear functions. By the lemma, if $n \geq 2$, then the polynomials $P$ and $Q$ have a common divisor that is not identically constant. The contradiction thus obtained shows that if $n \geq 2$, then the number of linear particular integrals of system (1) is less than $3n$.

For a given positive integer $n \geq 2$, system (1) with

$$P(x, y) \equiv P(x) = x \prod_{k=0}^{n-2} \left( x - \exp \frac{2\pi ik}{n-1} \right), \quad Q(x, y) \equiv Q(y) = y \prod_{k=0}^{n-2} \left( y - \exp \frac{2\pi ik}{n-1} \right)$$

admits not only the $2n$ particular integrals

$$x = 0, \quad y = 0, \quad x = \exp(2\pi ik/(n - 1)), \quad y = \exp(2\pi ik/(n - 1)), \quad k = 0, \ldots, n - 2,$$

but also the particular integrals

$$y = x \exp \frac{2\pi ij}{n - 1}, \quad j = 0, \ldots, n - 2.$$  

(5)

Indeed, the function (5) is a solution of the differential equation corresponding to system (1) with right-hand sides (4) if and only if the identity

$$P(x) \exp \frac{2\pi ij}{n - 1} \equiv Q \left( x \exp \frac{2\pi ij}{n - 1} \right)$$

holds for all $x$. By cancelling $x \exp(2\pi ij/(n - 1))$, from relations (6) and (4), we obtain the identity

$$\prod_{k=0}^{n-2} \left( x - \exp \frac{2\pi ik}{n-1} \right) \equiv \prod_{k=0}^{n-2} \left( x \exp \frac{2\pi ik}{n-1} - \exp \frac{2\pi ik}{n-1} \right).$$

(7)

By extracting $\exp(2\pi ij/(n - 1))$ from each factor on the right-hand side in identity (7) and by using the relation $(\exp(2\pi ij/(n - 1)))^{n-1} = 1$, we obtain

$$\prod_{k=0}^{n-2} \left( x - \exp \frac{2\pi ik}{n-1} \right) \equiv \prod_{k=0}^{n-2} \left( x - \exp \frac{2\pi i}{n-1}(k - j) \right), \quad j = 0, \ldots, n - 2.$$  

(8)

Let us show that identity (8) holds for all $x$. Since the respective factors on the left- and right-hand sides in identity (8) coincide for $j = 0$, we see that the identity holds for all $x$ if $j = 0$. Therefore,