ANALYTICAL AND NUMERICAL STUDY
OF GENERALIZED CAUCHY PROBLEMS
OCcurring IN GAS DYNAMICS

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The existence and uniqueness theorems for solutions in the class of analytic functions were proved for
two generalized Cauchy problems with conditions imposed on two surfaces. Using the theorems proved,
implicit difference schemes are constructed for the numerical solution. A program was designed, and
the corresponding numerical calculations were performed.

Keywords: gas dynamics, shock waves, boundary problem, existence theorem, difference scheme.

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Introduction. Mathematical description of ideal gas flows with shock waves [1] reduces to an initial-
boundary problem of special form for a system of quasilinear partial differential equations, which is called the
generalized Cauchy problem (GCP). Unlike in the traditional formulation of the Cauchy problem, in the GCP,
the initial-boundary conditions are given on two or more, rather than one, surfaces. In contrast to the mixed
problem, in the GCP the number of initial boundary-value conditions is equal to the number of unknown functions.
This problem was first considered in a study [2] of the compatibility of systems of partial differential equations.
The general existence and uniqueness theorem for solutions of the GCP is proved in [3]. A detailed survey of the
literature devoted to the GCP is given in [4].

A number of gas-dynamic problems which, in terms of the theory of partial differential equations, are GCPs
with initial and boundary conditions on two surfaces were studied by Teshukov [5–7]. For all problems consid-
ered in [5–7], the existence and uniqueness theorems for piecewise-analytical solutions were proved. In addition,
Kazakov [8], considering the gas-dynamic problem of shock-wave propagation from the axis or center of symmetry,
proved the existence and uniqueness of solutions of a GCP with a singularity.

In the present paper, the existence and uniqueness theorems for solutions in the class of analytic functions
are proved for two GCPs with initial and boundary conditions on two surfaces. The proven theorems are used to
construct implicit difference schemes for numerical solution of the problems. The systems of difference equations
reduce to systems of linear algebraic equations (SLAEs) with tridiagonal matrices. A program was designed which
was tested on model examples, and corresponding numerical calculations were performed.

1. Formulation of the Problem. We consider the system of gas-dynamic equations for an ideal polytropic
gas with the equation of state

\[ p = A^2(S)\rho^{\gamma}/\gamma, \]

where \( p \) is the pressure, \( S \) is the entropy, \( \rho \) is the density, and \( \gamma = \text{const} > 1 \) is the polytropic exponent of the gas.
Below, we investigate plane symmetric (\( \nu = 0 \)), cylindrically symmetric (\( \nu = 1 \)) or spherically symmetric (\( \nu = 2 \))
flows that depend on time \( t \) and the distance from the coordinate origin and the axis or center of symmetry \( r \).
The desired functions \( U = U(t,r) \) are functions \( U = (\sigma, u, s) \), where \( \sigma = \rho^{(\gamma-1)/2} \), \( u \) is the gas velocity, and \( s \) is a
function $A(S)$. The speed of sound is given by the relation $c = σs$, and the system of gas-dynamic equations has the form

$$σ_t + uσ_r + \frac{γ - 1}{2} σ (u_r + νu_τ) = 0,$$

$$u_t + \frac{2}{γ - 1} στ^2 σ_r + uu_r + \frac{2}{γ} στ^2 ss_r = 0,$$

$$s_t + us_r = 0.$$  \hspace{1cm} (1)

Suppose that in a neighborhood of the point $t = 0, r = 0$ system (1) has a known solution that is analytical in the variables $t$ and $r$. (This solution will be called background flow.) It is assumed that $u|_{t=r=0} < 0$, $σ|_{t=r=0} > 0$, and $s|_{t=r=0} > 0$. In addition, at $r = 0$, the gas velocity $u = 0$, so that there is a strong discontinuity of the solutions — a shock wave (SW), whose front is unknown and determined by solving the problem. A solution of system (1) behind the SW front is constructed. For $ν = 0$, the problem describes the reflection of the SW from the solid wall, and for $ν = 1$ and 2, it describes SW propagation from the axis or center of symmetry.

This gas-dynamic problem is reduced to a GCP by changing the independent and dependent variables in (1). For this, the independent variables $r$ and $t$ are first replaced by $x$ and $y$: $r = φ(x)$ and $t = y + x$. The Jacobian of the transformation is $J = φ'(x)$. The function $r = φ(t)$ defining the SW trajectory is unknown, but we know that $φ(0) = 0$. The main objective of the change is to transform the axis $r = 0$ to the axis $x = 0$ and the SW front to the $y = 0$ axis.

Before introducing new unknown functions, we denote the desired solution in the region between the SW front and the axis (center) of symmetry by $U$ and the background flow by $U^1$ and rewrite the Hugoniot conditions for the SW (i.e., on the axis $y = 0$) in equivalent form (which is possible by virtue of the determinacy theorem [1]):

$$D|_{y=0} = D^*(U^1, u)|_{y=0}, \quad σ|_{y=0} = σ^*(U^1, u)|_{y=0}, \quad s|_{y=0} = s^*(U^1, u)|_{y=0}. \hspace{1cm} (2)$$

The expression for $D^*$, $σ^*$, and $s^*$ are not given here for space reasons.

As noted above, $u|_{x=0} = 0$; therefore the quantities $D_0 = D|_{x=y=0}$, $σ_{00} = σ|_{x=y=0}$, and $s_{00} = s|_{x=y=0}$ are uniquely determined from (2). In this case, $c_{00} = s_{00}σ_{00} > 0$ and $D_0 > 0$. Consequently, the replacement at the point $t = 0, r = 0$ is nondegenerate, and if the functions $φ(x)$ are analytic, it will be nondegenerate in a neighborhood of the coordinate origin. We introduce new unknown functions $u'$, $v$, $z$, and $w$:

$$u' = u, \quad v = σ - σ^*, \quad w = φ, \quad z = s - s^*.$$  \hspace{1cm} (3)

After a series of transformations (rather lengthy), system (1) in the new variables becomes

$$u_x = \frac{1 - M_0^2}{1 + θM_0} u_y + \frac{M_0(1 + θ)}{1 + θM_0} v_x - \frac{ν}{1 + θM_0} u x + Y_1,$$

$$v_y = \frac{M_0(θ - 1)}{1 + θM_0} u_y + \frac{1}{1 + θM_0} v_x + \frac{νθ}{(1 + θM_0)(1 + θ)} u x + Y_2,$$

$$w_x = D^*|_{y=0}, \quad z_y = Y_3.$$  \hspace{1cm} (3)

Here the prime at the unknown function $u$ is omitted to simplify the notation, $M_0 = D_0/(σ_{00}σ_{00})$ (according to the Zemplén’s theorem, $0 < M_0 < 1$); the constant $θ$ arises from the differentiation of the Hugoniot conditions in the form (2) with respect to $u$ (it can be shown that $θ > 0$), and $Y_1$, $Y_2$, and $Y_3$ are functions that depend on the independent variables, unknown functions, and their first derivatives; moreover, the dependence on the derivatives is linear, and their coefficients vanish at the point $x = 0, y = 0$. In other words, system (1) is quasilinear, and in the coefficients of the derivatives, the main parts are isolated. The form of the functions $Y_1$, $Y_2$, and $Y_3$ is not given for space reasons.