Algorithmic Randomness and Splitting of Supermartingales\(^1\)

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Abstract—Randomness in the sense of Martin-L"of can be defined in terms of lower semicomputable supermartingales. We show that such a supermartingale cannot be replaced by a pair of supermartingales that bet only on even bits (the first one) and on odd bits (the second one) knowing all the preceding bits.

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1. RANDOMNESS AND LOWER SEMICOMPUTABLE SUPERMARTINGALES

The notion of algorithmic randomness (Martin-L"of randomness) for an infinite sequence of zeros and ones (with respect to a uniform Bernoulli distribution and independent trials) can be defined using supermartingales. In this context, a supermartingale is a nonnegative real-valued function \(m\) on binary strings such that

\[
m(x) \geq \frac{m(x0) + m(x1)}{2}
\]

for all strings \(x\). Any supermartingale corresponds to a strategy in the following game. In the beginning we have some initial capital \((m(\Lambda), \text{where } \Lambda \text{ is the empty string})\). Before each round, we put part of the money on zero, some other part on one, and throw away the rest. Then the next random bit of the sequence is generated, the correct stake is doubled, and the incorrect one is lost. In these terms, \(m(x)\) is the capital after bit string \(x\) appears. (If the option to throw away a portion of money is not used, then the inequality becomes an equality and the function \(m\) is a martingale.)

We say that a supermartingale \(m\) wins on an infinite sequence \(\omega\) if the values of \(m\) on the initial segments of \(\omega\) are unbounded. The set of all sequences where \(m\) wins is called its winning set.

Algorithmic probability theory often uses lower semicomputable supermartingales. This means that for each \(x\) the value \(m(x)\) is a limit of a nondecreasing sequence of nonnegative rational numbers \(M(x, 0), M(x, 1), \ldots\), and the mapping \((x, n) \mapsto M(x, n)\) is computable.

The class of all lower semicomputable supermartingales has a maximal element (up to a constant factor). Its winning set contains winning sets of all lower semicomputable supermartingales; this set is called the set of nonrandom sequences. The complement of this set is the set of random sequences.

This definition is equivalent to the standard definition given by Martin-L"of (e. g., see [1]); sometimes, it is called the criterion of Martin-L"of randomness in terms of supermartingales.

\(^1\) This work was done by Andrei Muchnik (1958–2007) in 2003. Soon after that, A. Chernov wrote a draft version of the paper based on his talk with Muchnik. Then Muchnik looked it through; he planned to edit the text but did not manage to do it. The present text was prepared by Chernov and A. Shen in 2007–2008; they are responsible for any possible errors and inaccuracies. A draft version of the paper was published as arXiv:0807.3156.

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Remark. Some authors also consider a larger class of “computably random” sequences that is obtained if lower semicomputable supermartingales are replaced by computable martingales (for simplicity, one can take rational-valued martingales). There is no maximal computable martingale, and a sequence is called computably random if any computable martingale is bounded on its initial segments. However, we keep the Martin-Löf definition as the main one; in the rest of the paper, “random” without additional adjectives means “Martin-Löf random.”

2. ODD/EVEN DECOMPOSITION OF MARTINGALES

One can observe that for computable martingales it is sufficient to separately consider bets on odd steps only and bets on even steps only to get the definition of randomness. We say that a (super)martingale bets only on even steps (or, in other words, does not bet on odd steps) if \( m(x) = m(x0) = m(x1) \) for all strings \( x \) of even length. (For instance, a martingale does not bet at the third step if after the third coin tossing the capital does not change, that is, if \( m(x0) = m(x1) = m(x) \) for every \( x \) of length 2. In terms of the game, this means that one half of the capital is put on 0 and the other half is put on 1, and in any case the capital remains the same.) Similarly, a (super)martingale bets only on odd steps if \( m(x0) = m(x1) = m(x) \) for every \( x \) of odd length.

Now let us give a precise formulation of the observation above: For any computable martingale \( t \) there exist two computable martingales \( t_0 \) and \( t_1 \), such that \( t_0 \) bets only on even steps and \( t_1 \) bets only on odd steps and the following holds: if \( t \) wins on some sequence \( \omega \), then either \( t_0 \) or \( t_1 \) wins on \( \omega \).

This implies that the winning set of \( t \) is contained in the union of the winning sets of \( t_0 \) and \( t_1 \).

Proof. Adding a constant if necessary, we may assume that \( t \) is strictly positive everywhere. Then we construct two martingales, \( t_0 \) and \( t_1 \), as follows: on even steps, \( t_0 \) divides its capital between zero and one in the same proportion as \( t \) does, and \( t_1 \) does not bet (puts equal stakes on zero and one); on odd steps, \( t_0 \) and \( t_1 \) change the roles. Then the gain of \( t \) (the current capital divided by the initial capital) equals to the product of the gains of \( t_0 \) and \( t_1 \). Therefore, if both \( t_0 \) and \( t_1 \) are bounded on prefixes of some sequence, so is \( t \).

In other words, defining randomness with respect to computable martingales, we may restrict ourselves to martingales that bet on every other step (on even or odd steps only). A similar statement is true for the Mises–Church definition of randomness (that uses selection rules; see [1]): it suffices to split a selection rule into two rules; one selects only even terms, and the other selects only odd terms.

It turns out, however, that for lower semicomputable supermartingales (and Martin-Löf random sequences) the analogous statement is wrong, and this is the main result of the paper.

Theorem 1. There exists a Martin-Löf nonrandom sequence \( \omega \) such that no supermartingale betting every other step (on even steps or on odd steps) wins on \( \omega \).

This result will be proved in Sections 3–5. Now let us point out its relations to the van Lambalgen theorem on randomness of pairs.

Any bit sequence \( \alpha \) can be split into two subsequences of even and odd terms, \( \alpha_0 \) and \( \alpha_1 \). It is easy to see that for a (Martin-Löf) random sequence \( \alpha \), both sequences \( \alpha_0 \) and \( \alpha_1 \) are random. However, the converse statement is not true (a trivial counterexample: even if \( \alpha_0 = \alpha_1 \) is random, the sequence with doubled bits is not).

A well-known theorem of van Lambalgen [2] gives a necessary and sufficient condition for the randomness of \( \alpha \): it is random if and only if \( \alpha_0 \) is random with the \( \alpha_1 \)-oracle and \( \alpha_1 \) is random with the \( \alpha_0 \)-oracle. (Informally, this means that even a player that can see any elements of \( \alpha_1 \) before betting on \( \alpha_0 \) and vice versa cannot win on any of these two sequences.) It would be natural to conjecture that there is no need to “look ahead” (choosing the stakes on the \((2n-1)\)st step, one