CODING THEORY

Analysis of the Relation between Properties of LDPC Codes and the Tanner Graph

V. V. Zyablov and P. S. Rybin

Kharkevich Institute for Information Transmission Problems,
Russian Academy of Sciences, Moscow
zyablov@iitp.ru prybin@iitp.ru

Received October 27, 2011; in final form, July 12, 2012

Abstract—A new method for estimating the number of errors guaranteed to be corrected by a low-density parity-check code is proposed. The method is obtained by analyzing edges with special properties of an appropriate Tanner graph. In this paper we consider binary LDPC codes with constituent single-parity-check and Hamming codes and an iterative decoding algorithm. Numerical results obtained for the proposed lower bound exceed similar results for the best previously known lower bounds.

DOI: 10.1134/S0032946012040011

1. INTRODUCTION

Error-correcting capabilities of Gallager’s low-density parity-check (LDPC) codes [1] for a binary symmetric channel (BSC) were studied in [2], where it was shown that there exists Gallager’s LDPC code capable of correcting a linear portion of errors, with decoding complexity $O(n \log n)$, where $n$ is the LDPC code length. Then in [3] a new (easier-to-compute) analytical lower bound on the error-correcting capabilities of the decoder of Gallager’s LDPC code was obtained by using combinatorial methods, but numerical results in most cases did not exceed those obtained by an old lower bound from [2]. It should be noted that the decoding algorithm considered in [3] differs from the algorithm described in [2]. In this paper we consider an algorithm similar to that from [2].

An LDPC code with a constituent Hamming code (H-LDPC code) was considered in [4]. The code distance and “soft” decoding of H-LDPC codes were then investigated in [5,6]. It was shown in [7] that the ensemble of H-LDPC codes contains such codes that the minimum code distance almost reaches the Gilbert–Varshamov bound. Then a result similar to that from [2] was first obtained for an H-LDPC code in [8] by generalization of methods developed in [2]. After that, using generalized methods from [8], similar results were obtained in [9] for a $q$-ary LDPC code used over a $q$-ary symmetric channel, and in [10] for generalized LDPC codes used over a binary erasure channel. The authors of [11] obtained a novel lower bound on the fraction of guaranteed corrected errors for binary codes on graphs used over a BSC. Numerical results for the lower bound [11] for H-LDPC codes exceed similar results for the best previously known lower bounds on error-correcting capabilities of H-LDPC codes.

In this paper we consider the same Gallager’s LDPC codes and H-LDPC codes and iterative low-complex hard-decision decoding algorithms as in [2,11]. We obtain a new lower bound on the fraction of guaranteed corrected errors by using methods developed in [2] and generalized in [8]. The main difference is in our approach. We consider edges with special properties of the corresponding Tanner graph instead of the number of constituent codes with unsatisfied checks as was done in [2,11]. Numerical computation for various choices of LDPC codes shows that the
proposed lower bound gives better results than the best previously known lower bounds obtained in [2,11].

Some of ideas and methods used in this paper were first introduced and developed in [9].

2. CONSTRUCTION AND PROPERTIES OF GALLAGER’S LDPC CODES AND H-LDPC CODES

Consider the construction of a parity-check matrix $H$ of a generalized LDPC code with a constituent code having a parity-check matrix $H_0$. Let $H_b$ denote a block diagonal matrix with $b$ copies of the constituent parity-check matrix $H_0$ on the main diagonal, i.e.,

$$H_b = \begin{pmatrix}
H_0 & 0 & \ldots & 0 \\
0 & H_0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & H_0
\end{pmatrix},$$

where $b$ is very large. If the matrix $H_0$ is of size $m_0 \times n_0$, then $H_b$ is of size $bm_0 \times bn_0$. Let $\pi(H_b)$ denote a random column permutation of $H_b$. Then the matrix

$$H = \begin{pmatrix}
\pi_1(H_b) \\
\pi_2(H_b) \\
\vdots \\
\pi_\ell(H_b)
\end{pmatrix} = \begin{pmatrix}
H_1 \\
H_2 \\
\vdots \\
H_\ell
\end{pmatrix},$$

constructed using $\ell > 2$ such permutation as layers is a sparse parity-check matrix $H$ of size $\ell b m_0 \times b n_0$, which defines an ensemble of generalized LDPC codes of length $n = b n_0$, where $n \gg n_0$, with a given parity-check matrix $H_0$ of a constituent code. Denote this ensemble by $\mathcal{E}_{H_0}(n_0, \ell, b)$.

**Definition 1.** For a given constituent code with parity-check matrix $H_0$, elements of ensemble $\mathcal{E}_{H_0}(n_0, \ell, b)$ are obtained by independently and equiprobably sampling the permutations $\pi_l$, $l = 1, 2, \ldots, \ell$.

**Remark 1.** It should be noted that $\mathcal{E}_{H_0}(n_0, \ell, b)$ does not define the ensemble of all generalized LDPC codes but defines the ensemble of generalized LDPC codes with a given constituent code. In this paper we will consider only single-parity-check (SPC) and Hamming codes as constituent codes. If necessary, we will specify a constituent code explicitly; otherwise, the ensemble $\mathcal{E}_{H_0}(n_0, \ell, b)$ should be considered as an ensemble of LDPC codes with some given constituent code.

For historical reasons, we refer to an LDPC code with a constituent SPC code as Gallager’s LDPC code, and to an LDPC code with a constituent Hamming code, as an H-LDPC code.

The rate $R$ of a code from $\mathcal{E}_{H_0}(n_0, \ell, b)$ is lower bounded by

$$R \geq 1 - \ell(1 - R_0),$$

where $R_0$ is the rate of the constituent code. The equality is achieved if and only if $H$ is a full-rank matrix.

As follows from the construction, a generalized LDPC code from $\mathcal{E}_{H_0}(n_0, \ell, b)$ has $n = b n_0$ code symbols, which are distributed between $\ell b$ constituent codes ($b$ in each layer) with the constituent matrix $H_0$. Such codes can be represented by a bipartite Tanner graph [12] $G = (V_1 : V_2, E)$ with $n = b n_0$ variable nodes $V_1$ and $\ell b$ constraint nodes $V_2$ (Fig. 1). Each constraint node comprises $m_0$ parity-check constraints specified by rows of the corresponding parity-check matrix $H_0$. 