ON THE LOCAL BEHAVIOR OF MAPPINGS WITH UNBOUNDED QUASICONFORMALITY COEFFICIENT

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Abstract: We study space mappings more general than the mappings with bounded distortion in the sense of Reshetnyak. We consider questions related to the local behavior of mappings differentiable almost everywhere, possessing Properties $N$, $N^{-1}$, $ACP$, and $ACP^{-1}$, and such that quasiconformality coefficient satisfies a certain restriction on growth. We show that the value of a mapping satisfying these requirements on an arbitrary neighborhood of an essential singularity can be greater in absolute value than the logarithm of the inverse radius of the ball raised to an arbitrary positive power.

Keywords: mapping of bounded and finite distortion, modulus of a family of curves

1. Introduction

Denote the Lebesgue measure on $\mathbb{R}^n$ by $m$, the Euclidean distance between two sets $A, B \subset \mathbb{R}^n$ by

$$\text{dist}(A, B) = \inf_{x \in A, y \in B} |x - y|,$$

the (standard) inner product of $x, y \in \mathbb{R}^n$ by $(x, y)$, and the Euclidean diameter of $A \subset \mathbb{R}^n$ by $\text{diam} A$. Put

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$$

and $\mathbb{B}^n := B(0, 1)$, as well as

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$$

and $\mathbb{S}^{n-1} := S(0, 1)$. Denote the area of the sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ by $\omega_{n-1}$ and the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ by $\Omega_n$.

Given a domain $D$ in $\mathbb{R}^n$ with $n \geq 2$, the record $f : D \to \mathbb{R}^n$ means that $f$ is a continuous mapping on $D$. As usual, we write $f \in W^{1,n}_1(D)$ whenever all coordinate functions $f = (f_1, \ldots, f_n)$ possess generalized first-order partial derivatives that are locally integrable on $D$ to the power $n$. A mapping $f : D \to \mathbb{R}^n$ is called discrete whenever the preimage $f^{-1}(y)$ of every point $y \in \mathbb{R}^n$ consists of isolated points, and open whenever the image of every open subset $U \subset D$ is open in $\mathbb{R}^n$. Say that a mapping $f : D \to \mathbb{R}^n$ possesses the Luzin $N$-property or, simply, $N$-property whenever the conditions $m(E) = 0$ and $E \subset D$ imply that $m(f(E)) = 0$. Similarly, say that a mapping $f : D \to \mathbb{R}^n$ possesses the $N^{-1}$-property whenever the conditions $m(E) = 0$ and $E \subset \mathbb{R}^n$ imply that $m(f^{-1}(E)) = 0$ where, as usual, $f^{-1}(E)$ stands for the full preimage of $E$ under $f$.

Recall that $f : D \to \mathbb{R}^n$ is called a mapping with bounded distortion if the following hold:

1. $f \in W^{1,n}_1$;
2. the Jacobian $J(x, f) := \det f'(x)$ of $f$ at $x \in D$ is of the same sign a.e. in $D$,
3. $\|f'(x)\|^n \leq K|J(x, f)|$ for a.e. $x \in D$ and some constant $K < \infty$, where, as usual,

$$\|f'(x)\| := \sup_{h \in \mathbb{R}^n, |h| = 1} |f'(x)h|$$

(see [1, Chapter I, § 3] for instance or Definition 2.1 of [2, Chapter I, Section 2]).

Reshetnyak pioneered the thorough study of space mappings with bounded distortion. His articles show in particular that the mappings with bounded distortion are discrete and open (see Theorems 6.3...
and 6.4 of [1, Chapter II]), almost everywhere differentiable (see Theorem 4 of [3]), and possess the Luzin $N$-property (see [1, Chapter II, Theorem 6.2]). On the other hand, Bojarski and Iwaniec established that the mappings with bounded distortion possess the $N^{-1}$-property (see Theorem 8.1 of [4]). Recall that a point $x_0$ of the boundary $\partial D$ of a domain $D$ is called a removable singularity of $f$ whenever the some finite limit $\lim_{x \to x_0} f(x)$ exists. If $f(x) \to \infty$ as $x \to x_0$ then $x_0$ is called a pole. An isolated point $x_0$ of $\partial D$ is called an essential singularity of $f : D \to \mathbb{R}^n$ if neither finite nor infinite limit exists as $x \to x_0$. Väisälä proved the following (see Theorem 4.2 of [5]):

**Proposition 1.** Consider $b \in D$ and a mapping $f : D \setminus \{b\} \to \mathbb{R}^n$ with bounded distortion. Assume that there exists $\delta > 0$ such that

$$|f(x)| \leq C|x - b|^{-p}$$

for all $x \in B(b, \delta) \setminus \{b\}$, where $p > 0$ and $C > 0$ are some constants. Then $b$ is either a pole or a removable singularity of $f$.

The goal of this article is to prove Proposition 1 for more general classes of mappings which include the class of mappings with bounded distortion. Martio, Ryazanov, Srebro, and Yakubov (see [6] or [7, Section 8] for instance) introduced a.e. differentiable mappings of class $\text{ACP} \cap \text{ACP}^{-1}$ with Luzin’s $N$- and $N^{-1}$-properties, called mappings with finite length distortion. Subsequently these mappings were studied in connection with various problems. They are important primarily in the thorough study of general theories of mappings with finite distortion (see [8, Chapter 20; 9, Chapter 6; 10–14] for instance). In particular, their study has some useful applications, for instance, Sobolev classes (see the last section of this article).

Refer as a curve $\gamma$ to a continuous mapping $\gamma : [a, b] \to \mathbb{R}^n$ of the segment $[a, b]$ (or the open interval $(a, b)$ or the half-open intervals $[a, b)$ or $(a, b]$) into $\mathbb{R}^n$. By a family $\Gamma$ of curves we understand a fixed collection of curves $\gamma$, and put $f(\Gamma) = \{f \circ \gamma \mid \gamma \in \Gamma\}$. The following definition appears, for instance, in [15, Chapter I, Section 1–6]: a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called admissible for a family $\Gamma$ of curves $\gamma$ in $\mathbb{R}^n$ whenever the line integral of the first kind of $\rho$ over every (locally rectifiable) curve $\gamma \in \Gamma$ satisfies $\int_\gamma \rho(x)|dx| \geq 1$. In this case we write $\rho \in \text{adm} \Gamma$. Refer as the modulus of $\Gamma$ to

$$M(\Gamma) = \inf_{\rho \in \text{adm} \Gamma} \int_D \rho^n dm(x).$$

The properties of the modulus are reminiscent of the Lebesgue measure $m$ on $\mathbb{R}^n$. Namely, the modulus of the empty family of curves equals zero, $M(\emptyset) = 0$; the modulus is monotone with respect to families of curves $\Gamma_1$ and $\Gamma_2$, meaning that $\Gamma_1 \subset \Gamma_2 \Rightarrow M(\Gamma_1) \leq M(\Gamma_2)$, and semiadditive:

$$M\left(\bigcup_{i=1}^{\infty} \Gamma_i\right) \leq \sum_{i=1}^{\infty} M(\Gamma_i)$$

(see Theorem 6.2 of [15]). Say that some property holds for almost every curve in $D$ if it holds for all curves in $D$ except for some family whose modulus equals zero.

Take an open interval $\Delta \subset \mathbb{R}$ and a locally rectifiable curve $\gamma : \Delta \to \mathbb{R}^n$. It is obvious that there exists a unique nondecreasing length function $l_\gamma : \Delta \to \mathbb{R}_+ \subset \mathbb{R}$ with $l_\gamma(t_0) = 0$ for $t_0 \in \Delta$ such that $l_\gamma(t)$ equals the length of the arc $\gamma|_{[t_0, t]}$ of $\gamma$ if $t > t_0$, and $-l_\gamma|_{[t, t_0]}$ if $t < t_0$ and $t \in \Delta$. Consider a continuous mapping $g : |\gamma| \to \mathbb{R}^n$, where $|\gamma| = \gamma(\Delta) \subset \mathbb{R}^n$. Assume that the curve $\tilde{\gamma} = g \circ \gamma$ is also locally rectifiable. It is obvious then that there exists a unique nondecreasing function $L_{\gamma,g} : \Delta_\gamma \to \Delta_\gamma$ such that $L_{\gamma,g}(l_\gamma(t)) = l_\gamma(t)$ for all $t \in \Delta$. The curve $\gamma \in D$ is called a (total) lifting of the curve $\tilde{\gamma} \in \mathbb{R}^n$ under the mapping $f : D \to \mathbb{R}^n$ whenever $\tilde{\gamma} = f \circ \gamma$.

Say that a mapping $f : D \to \mathbb{R}^n$ is of class $\text{ACP}$ in a domain $D$, and write $f \in \text{ACP}$, if for every a.e. curve $\gamma$ in $D$ the curve $\tilde{\gamma} = f \circ \gamma$ is locally rectifiable and the length function $L_{\gamma,f}$ introduced above is absolutely continuous on all closed intervals in $\Delta_\gamma$. Assume that $f : D \to \mathbb{R}^n$ is a discrete mapping.