PROPERMUTABLE CHARACTERIZATIONS OF
FINITE SOLUBLE PST-GROUPS AND PT-GROUPS
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Abstract: Let $H$ and $X$ be subgroups of a group $G$. We say that a subgroup $H$ is $X$-propermutable
in $G$ provided that there is a subgroup $B$ of $G$ such that $G = N_{G}(H)B$ and $H$ $X$-permutes (in the sense
of [1]) with all subgroups of $B$. In this paper we analyze the influence of $X$-propermutable subgroups
on the structure of a finite group $G$. In particular, it is proved that $G$ is a soluble PST-group if and
only if all Hall subgroups and all maximal subgroups of every Hall subgroup of $G$ are $X$-propermutable
in $G$, where $X = Z_{\infty}(G)$.

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1. Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. The symbol $\pi(n)$
stands for the set of all primes dividing the number $n$; $\pi(G) = \pi(|G|)$. The symbol $G^{\pi}$ denotes the
nilpotent residual of $G$, i.e. the smallest normal subgroup of $G$ with nilpotent quotient.

Let $A$, $B$, and $X$ be subgroups of $G$. Then $A$ is said to permute with $B$ if $AB = BA$ and $X$-permute
with $B$ [1] if $ABx = BxA$ for at least one $x \in X$.

The subgroup $A$ is said to be permutable (S-permutable) in $G$ if $A$ permutes with all subgroups (with
all Sylow subgroups, respectively) of $G$. A group $G$ is called a PT-group if permutability is a transitive
relation on $G$; i.e., every permutable subgroup of a permutable subgroup of $G$ is permutable in $G$.
A group $G$ is called a PST-group if S-permutability is a transitive relation on $G$.

As well as T-groups, PT-groups, and PST-groups possess many interesting properties (see [2,
Chapter 2]). The descriptions of PT-groups and PST-groups were firstly obtained by Zacher [3] and
Agrawal [4], for the soluble case; and by Robinson in [5], for the general case. Nevertheless, in the further
publications, the authors (see [2] or the recent papers [6–16]) have found out many other descriptions of
soluble PT-groups and PST-groups.

In this paper we give some new characterizations of soluble PST-groups and PT-groups on the basis
of the following

Definition 1.1. Let $H$ and $X$ be subgroups of $G$. We say that $H$ is $X$-propermutable in $G$ provided
that there is a subgroup $B$ of $G$ such that $G = N_{G}(H)B$ and $H$ $X$-permutes with all subgroups of $B$.

If $X = 1$ in this definition, then $H$ is said to be propermutable in $G$. We say also that $H$ is completely
propermutable in $G$ (in this connection, see Question 18.91 in [17]) if $H$ is propermutable in every
subgroup of $G$ including $H$.

Our main goal here is to prove the following

Theorem A. Let $X = Z_{\infty}(G)$. Then $G$ is a soluble PST-group if and only if all Hall subgroups
of $G$ and all maximal subgroups of every Hall subgroup of $G$ are $X$-propermutable in $G$.

The proof of Theorem A consists of many steps and the following three useful results cover the main
stages of it.

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**Proposition 1.2.** Let $X = F(G)$ be the Fitting subgroup of $G$ and let $H$ be a Hall $X$-permutable subgroup of $G$. If $p > q$ for all primes $p$ and $q$ such that $p$ divides $|H|$ and $q$ divides $|G : H|$, then $H$ is normal in $G$.

The subgroup $A$ of $G$ is said to be $X$-semipermutable in $G$ [18] if $G$ has a subgroup $B$ such that $G = AB$ and $A$ permutes with all subgroups of $B$.

**Example 1.3.** Let $p$ and $q$ be primes such that $q$ divides $p - 1$. Let $G = A \times B$, where $A$ is a nonabelian group of order $pq$ and $B$ is a group of order $p$. Let $H$ be a subgroup of order $q$ in $G$. Then, clearly, $H$ is completely propermutable in $G$ and $H$ is not $G$-semipermutable in $G$.

The following corollary of Proposition 1.2 is equivalent to Theorem 5.4 in [18].

**Corollary 1.4.** Let $X = F(G)$ and let $H$ be a Hall subgroup of $G$. Suppose that $H$ is $X$-semipermutable in $G$ and $p > q$ for all primes $p$ and $q$ such that $p$ divides $|H|$ and $q$ divides $|G : H|$. Then $H$ is normal in $G$.

**Corollary 1.5** (see [19, Theorem 3]). If a Sylow $p$-subgroup $P$ of $G$, where $p$ is the largest prime dividing $|G|$, is 1-semipermutable in $G$, then $P$ is normal in $G$.

**Proposition 1.6.** Let $X = F(G)$ be the Fitting subgroup of $G$. If every Sylow subgroup of $G$ is $X$-permutable in $G$, then $G$ is supersoluble.

**Corollary 1.7** (see [19, Theorem 5]). If every Sylow subgroup of $G$ is 1-semipermutable in $G$, then $G$ is supersoluble.

**Proposition 1.8.** Let $G$ be a supersoluble group, $X = F(G)$, and $\pi = \pi(G^{\mathfrak{p}})$. Suppose that every subgroup of $G$, which is either a subnormal $\pi$-subgroup of $G$ or, for some $p \in \pi$, a maximal subgroup of some Sylow $p$-subgroup of $G$, is $X$-permutable in $G$. Then $G^{\mathfrak{p}}$ is a Hall subgroup of $G$.

On the basis of Theorem A and Proposition 1.8 we also prove the following

**Theorem B.** A soluble group $G$ of odd order is a PT-group if and only if all Hall subgroups and all subnormal subgroups of $G$ are completely propermutable in $G$.

### 2. The Basic Lemmas

The next lemma is evident.

**Lemma 2.1.** Let $A$, $B$, and $X$ be subgroups of $G$ and let $N$ be a normal subgroup of $G$.
1. If $A$ permutes with $B$, then $AN/N$ permutes with $BN/N$ in $G/N$.
2. If $N \leq A$ and $A/N$ permutes with $BN/N$ in $G/N$, then $A$ permutes with $B$ in $G$.

**Lemma 2.2.** Let $H$ and $X$ be subgroups of $G$ and let $N$ be a normal subgroup of $G$.
1. If $H$ is $X$-permutable in $G$, then $HN/N$ is $(XN/N)$-permutable in $G/N$.
2. If $H$ is propermutable in $G$, then $H$ permutes with some Sylow $p$-subgroup of $G$ for every prime $p$ dividing $|G|$.
3. If $N \leq H$ and $H/N$ is $(XN/N)$-permutable in $G/N$, then $H$ is $X$-permutable in $G$.
4. If $H$ is propermutable in $G$, then $NH$ is propermutable in $G$.
5. If $H$ is completely propermutable in $G$, then $HN/N$ is completely propermutable in $G/N$.

**Proof.** (1) By hypothesis, there is a subgroup $B$ of $G$ such that $G = N_G(H)B$ and $H$ permutes with all subgroups of $B$. It is clear that


Let $K/N$ be a subgroup of $BN/N$. Then $K = (K \cap B)N$, and so $HN/N$ permutes with $K/N$ in $G/N$ by Lemma 2.1. Therefore $HN/N$ is $(XN/N)$-propermutable in $G/N$.

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