ON THE LINEAR LOGIC OF KNOWLEDGE AND TIME WITH INTRANSITIVE TIME RELATION

V. F. Yun

Abstract: In [1], there was introduced a linear multi-modal logic of knowledge and time with intransitive time relation as the set of formulas valid in frames of special kind. The article [2] proposed some calculus $AS_{LTK_r}$ that is connected with the class of these frames. In the present article we find a formula of the linear logic of knowledge and time which is underviable in $AS_{LTK_r}$.

DOI: 10.1134/S0037446615030209

Keywords: multi-modal logic, Kripke frame, axiomatization, completeness

Introduction

The present article deals with the multi-modal logic of knowledge and time with intransitive time relation. More exactly, in [1] the $LTK_r$-frames $\langle \bigcup_{n \in J} C^n, R_T, R_\sim, R_1, \ldots, R_k \rangle$ were considered with $R_T$-clusters of states $C^n$ intransitive on the clusters, with $R_T$ piecewise strongly connected and reflexive, and the multi-modal logic $LTK_r$ was defined as the set of formulas valid in all $LTK_r$-frames.

In [2], the study of the logic $LTK_r$ was continued and the question of its axiomatization was considered. The authors found the calculus $AS_{LTK_r}$ correct with respect to the class of all $LTK_r$-frames. Moreover, in [2] the completeness of the calculus with respect to the class of frames connected with $LTK_r$-frames was proved.

In this article, we find a formula underviable in $AS_{LTK_r}$ and valid in all $LTK_r$-frames. Thus, we prove the lack of completeness for $AS_{LTK_r}$ with respect to the class of all $LTK_r$-frames.

1. The Calculus $AS_{LTK_r}$ and the Correctness Theorem

Consider the modal language with modal operators $\Box_T, \Box_\sim, \Box, \ldots, \Box_k$. More exactly, consider the language consisting of the countable set of propositional variables $P$, the standard logical connectives, and modal operators $\Box_T, \Box_\sim, \Box, \ldots, \Box_k$. Here the formulas are defined as usual [3].

We will consider the frames $\langle W, R_T, R_\sim, R_1, \ldots, R_k \rangle$, and models of the form $\langle W, R_T, R_\sim, R_1, \ldots, R_k, V \rangle$, where $W$ is a nonempty set, $R_T, R_\sim, R_1, \ldots, R_k$ are binary relations on $W$, and $V$ is a valuation of variables, i.e. some mapping $V : P \to \mathcal{P}(W)$. The valuation $V$ can be extended in a standard way [3] to the set of formulas of the language under consideration. In particular, for every $x \in W$, we have $x \models_V p \iff x \in V(p)$ for every variable $p \in P$ and

$x \models_V \Box_T A \iff \forall y(xR_Ty \models y \models_V A)$,

$x \models_V \Box_\sim A \iff \forall y(xR_\sim y \models y \models_V A)$,

$x \models_V \Box i A \iff \forall y(xR_i y \models y \models_V A)$, $i \in \{1, \ldots, k\}$.

A formula $A$ is true in the model $M = \langle W, R_T, R_\sim, R_1, \ldots, R_k, V \rangle$ if $x \models_V A$ for every $x \in W$. We say that a formula $A$ is valid in a frame if it is true in every model based on the frame.

Definition 1. A frame $\langle W, R_T, R_\sim, R_1, \ldots, R_k \rangle$ is called an $LTK_r$-frame [2] provided that

(a) $W = \bigcup_{n \in J} C^n$, where $C^n \neq \emptyset$, $J = \{1, \ldots, L\}$, $L \in \mathbb{N}$, or $J = \mathbb{N}$;

The author was supported by the State Maintenance Program for the Leading Scientific Schools of the Russian Federation (Grant NSh–860.2014.1).


0037-4466/15/5603-0565 565
(b) \(xR_T y \iff \exists n \in J((x \in C^n \text{ and } y \in C^n) \text{ or } (x \in C^n \text{ and } y \in C^{n+1}));\)
(c) \(xR_{\sim} y \iff \exists n \in J(x \in C^n \text{ and } y \in C^n);\)
(d) \(R_T\) is an equivalence relation on each \(C^n\), i.e. such that \(xR_T y\) implies \(x \in C^n\) and \(y \in C^n\) for some \(n \in J\).

Recall that \(C_{R_T}\) is \((C_{R_{\sim}})\) in \(W\) is called a \(C_{R_T}\)-\(\bar{\text{cluster}}\) (a \(C_{R_{\sim}}\)-\(\bar{\text{cluster}}\)) if \(\forall w \forall z \in C_{R_T}(wR_T z \& zR_T w)\) and \(\forall z \in W \forall w \in C_{R_T}(wR_T z \& zR_T w)\) and \(\forall z \in W \forall w \in C_{R_T}(wR_T z \& zR_T w)\) respectively.

Thus, each set \(C^n\) is an \(R_T\)-\(\bar{\text{cluster}}\) (and an \(R_{\sim}\)-\(\bar{\text{cluster}}\)); i.e., for \(x \in C^n\) we have \(C^n = \{y \mid xR_T y\text{ and }yR_T x\}\).

Consider the calculus \(AS_{LTK_r}\) introduced in [2].

**The Axioms of** \(AS_{LTK_r}\). The tautologies of the classical propositional logic;
\[
\begin{align*}
L_{\Box T} & : \Box_T(\Box_T A \rightarrow B) \lor \Box_T(\Box_T B \rightarrow A); \\
K_{\Box_T} & : \Box_T(A \rightarrow B) \rightarrow (\Box_T A \rightarrow \Box_T B), \; \xi \in \{T, \sim, 1, \ldots, k\}; \\
L_{\Box_T} & : \Box_T A \rightarrow A, \; \xi \in \{T, \sim, 1, \ldots, k\}; \\
K_{\Box_T} & : \Box_T A \rightarrow \Box_T A, \; \xi \in \{\sim, 1, \ldots, k\}; \\
L_{\Box_T} & : \Box_T A \rightarrow \Box_T A, \; 1 \leq i \leq k; \\
L_{\Box_T} & : \Box_T A \rightarrow \Box_T A, \; 1 \leq i \leq k; \\
A_T & : (\Box_T A \& \Box_T B \& \Box_T(\neg A \& \Box_T B)) \rightarrow \Box_T B.
\end{align*}
\]

**The Inference Rules.**
\[
\begin{align*}
\text{MP} & : \frac{A, A \rightarrow B}{B}, \; \text{Nec} : \frac{A}{\Box_T A}.
\end{align*}
\]

Here and below, \(\Diamond_T\) is the abbreviation of \(\neg \Box_T \neg\) (\(\xi \in \{T, \sim, 1, \ldots, k\}\)).

Let us introduce a class of frames containing the class of \(LTK_r\)-frames and prove that the calculus \(AS_{LTK_r}\) is correct with respect to this class.

**Definition 2.** Let us call a frame \(\langle W, R_T, R_{\sim}, R_1, \ldots, R_k \rangle\) an \(LTK_r\)-\(\bar{\text{frame}}\) if it satisfies the following conditions:

<table>
<thead>
<tr>
<th>PL(_{\Box_T}):</th>
<th>if (xR_T y) and (xR_T z) then (yR_T z) or (zR_T y);</th>
</tr>
</thead>
<tbody>
<tr>
<td>PT(_{\Box_T}):</td>
<td>the relations (R_T), (R_{\sim}), and (R_i) (1 (\leq i \leq k)) are reflexive;</td>
</tr>
<tr>
<td>P4(_{\Box_T}):</td>
<td>the relations (R_{\sim}) and (R_i) (1 (\leq i \leq k)) are transitive;</td>
</tr>
<tr>
<td>P5(_{\Box_T}):</td>
<td>the relations (R_{\sim}) and (R_i) (1 (\leq i \leq k)) are symmetric;</td>
</tr>
<tr>
<td>PM.1:</td>
<td>if (xR_{\sim} y) then (xR_T y);</td>
</tr>
<tr>
<td>PM.2:</td>
<td>if (xR_{\sim} y) then (xR_{\sim} y) for each (i \in {1, \ldots, k});</td>
</tr>
<tr>
<td>PAL:</td>
<td>if (xR_T yR_T z) and (x, y, z) belong to different (R_{\sim})-(\bar{\text{clusters}}) then (xR_T z) fails.</td>
</tr>
</tbody>
</table>

Refer to a model based on an \(LTK_r\)-frame as an \(LTK_r\)-\(\bar{\text{model}}\).

**Theorem 1.** If a formula \(A\) is derivable in \(AS_{LTK_r}\), then \(A\) is valid in every \(LTK_r\)-\(\bar{\text{frame}}\).

**Proof.** It suffices to prove that the axioms of \(AS_{LTK_r}\) are valid in every \(LTK_r\)-frame.

Prove the validity of \(L_{\Box_T}\). Suppose that the formula
\[
\Box_T(\Box_T A \rightarrow B) \lor \Box_T(\Box_T B \rightarrow A)
\]
is not valid. Then it is refuted in some \(LTK_r\)-\(\bar{\text{model}}\) \(\langle W, R_T, R_{\sim}, R_1, \ldots, R_k, V \rangle\); i.e., \(x \not\equiv_V \Box_T(\Box_T A \rightarrow B) \lor \Box_T(\Box_T B \rightarrow A)\) for some \(x \in W\). Then \(x \equiv_V \Box_T(\Box_T A \& \neg B) \& \Box_T(\Box_T B \& \neg A)\). Consequently, there exists \(y \in W\) such that \(xR_T y\) and \(y \equiv_V \Box_T A \& \neg B\) and there exists \(z \in W\) such that \(xR_T z\) and \(z \equiv_V \Box_T B \& \neg A\).

Since \(xR_T y\) and \(xR_T z\), by property \(PL_{\Box_T}\) we have \(yR_T z\) or \(zR_T y\). If \(yR_T z\) then \(z \equiv_V A\) because \(y \equiv_V \Box_T A\). This contradicts the fact that \(z \equiv_V \neg A\). The case of \(zR_T y\) is proved similarly.