QUASICONFORMAL EXTENSION OF QUASIMÖBIUS MAPPINGS OF JORDAN DOMAINS

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Abstract: We introduce the new class of Jordan arcs (curves) of bounded rotation which includes all arcs (curves) of bounded turning. We prove that if the boundary of a Jordan domain has bounded rotation everywhere but possibly one singular point then every quasimöbius embedding of this domain extends to a quasiconformal automorphism of the entire plane.

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1. Introduction

The problem of quasiconformal extension of a topological embedding \( f : M \rightarrow \mathbb{R}^2 \), where \( M \subset \mathbb{R}^2 \), to the entire plane was originally stated and completely solved in 1956 by Ahlfors and Beurling [1] in the case that \( M \) is a line. Rickman in 1966 proved in [2] the existence of a quasiconformal extension in the case that \( M \) is a Jordan arc of bounded turning, and established in [3] in 1968 the most general, though hard to check sufficient condition for continuing a homeomorphism of the boundaries of Jordan domains to a quasiconformal mapping of these domains, which in particular includes the requirement that the boundary mapping be quasimöbius, in modern terminology. In a series of articles [4–9] devoted to the problem of quasiconformal extension from Jordan domains, the original homeomorphism was assumed to be quasimöbius (a necessary condition for extension) and geometric conditions on the boundary of the domain were stated in terms of arcs of bounded turning. The results related to this problem are surveyed in [10, Chapter 4].

In this article we show (Theorem 3.2) that the weaker property of bounded rotation rather than the familiar property of bounded turning, introduced and studied in Section 2, is responsible for the existence of a quasiconformal extension. The central and most laborious point of the proof of our main theorem is Lemma 4.1 in which we study some topological properties of curves of bounded rotation everywhere but possibly one singular point.

2. Arcs of Bounded Turning

Denote by \( \sigma(\cdot, \cdot) \) the chordal metric on \( \mathbb{R}^2 \). Given \( a, b \in \mathbb{R}^2 \) with \( a \neq b \), the formula

\[
h_{(a,b)}(x, y) := \frac{\sigma(x, y)\sigma(a, b)}{\sigma(x, a)\sigma(y, b) + \sigma(y, a)\sigma(x, b)}
\]

(2.1)
determines the Möbius invariant metric on \( \mathbb{R}^2 \setminus \{a, b\} \) introduced in [11] in the Ptolemaic spaces and called the angular metric; i.e., \( h_{(a,b)}(x, y) = h_{(\mu(a), \mu(b))}(\mu(x), \mu(y)) \) for every Möbius automorphism \( \mu : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) for all \( a \neq b \) and \( x, y \in \mathbb{R}^2 \setminus \{a, b\} \). If \( a, b, x, y \in \mathbb{R}^2 \) then we can replace the chordal distance in (2.1) with the Euclidean distance. In particular, if the Möbius transformation \( \mu : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) maps \( a \) and \( b \) (the poles) into \( 0 \) and \( \infty \) respectively then

\[
h_{(a,b)}(x, y) = h_{(0,\infty)}(\mu(x), \mu(y)) = \frac{|\mu(x) - \mu(y)|}{|\mu(x)| + |\mu(y)|}.
\]

(2.2)
Observe that \( h_{(a,b)}(x,y) \leq 1 \) for all \( x,y \in \overline{R^2} \setminus \{a,b\} \), where the equality \( h_{(a,b)}(x,y)=1 \) amounts to the property that the points \( x, a, y, \) and \( b \) lie on the same circle and the pair \( a,b \) separates the pair \( x,y \). Elementary calculations justify

2.3. Proposition. For \( 0 < \varepsilon < 1 \) the open disk \( B_{(0,\infty)}(1,\varepsilon) := \{ z \in \mathbb{C} : h_{(0,\infty)}(1,z) < \varepsilon \} \) is a Jordan domain symmetric with respect to the real axis and the unit circle, and inscribed into the angular sector \( \{ z = re^{i\varphi} : |\varphi| \leq 2\sin^{-1}\varepsilon \} \).

The Möbius invariance of (2.1) implies

2.4. Proposition. For arbitrary \( a,b \in \overline{R^2} \) with \( a \neq b \) and \( x_0 \in \overline{R^2} \setminus \{a,b\} \) the open disk \( B_{(a,b)}(x_0,r) \) of radius \( r \in (0,1) \) is a Jordan domain and includes no continua separating the sphere \( \overline{R^2} \) between the poles \( a \) and \( b \).

2.5. Proposition. If \( 0 < \varepsilon < 1 \) for \( p > 1 \) and

\[
R > (1 + 2/\varepsilon)^{p/(p-1)}
\]

then for arbitrary \( c,d \in \overline{B}(0,R^{1/p}) \) and \( a,b \in \overline{R^2} \setminus B(0,R) \) the following hold:

\[
h_{(a,b)}(c,d) = h_{(c,d)}(a,b) < \varepsilon,
\]

\[
\overline{B}(0,R^{1/p}) \subset B_{(a,b)}(c,\varepsilon),
\]

\[
R^2 \setminus B(0,R) \subset B_{(c,d)}(a,\varepsilon).
\]

Moreover, for all pairs \( c^*,d^* \in \overline{R^2} \setminus B(0,R^{(p-1)/p}) \) and \( a^*,b^* \in \overline{B}(0,1) \) we have

\[
h_{(a^*,b^*)}(c^*,d^*) = h_{(c^*,d^*)}(a^*,b^*) < \varepsilon,
\]

\[
R^2 \setminus B(0,R^{(p-1)/p}) \subset B_{(a^*,b^*)}(c^*,\varepsilon),
\]

\[
\overline{B}(0,1) \subset B_{(c^*,d^*)}(a^*,\varepsilon).
\]

Proof. Since \( |c - d| \leq 2R^{1/p} \), \( |d - b| \geq R - R^{1/p} \), and \( |d - a| \geq R - R^{1/p} \), we have

\[
h_{(a,b)}(c,d) = h_{(c,d)}(a,b) = \frac{|a - b| \cdot |c - d|}{|a - c| \cdot |b - d| + |a - d| \cdot |b - c|}
\]

\[
\leq \frac{2R^{1/p}}{R^{1/p}(R^{(p-1)/p} - 1) |a - c| + |c - b|} \leq \frac{2}{R^{(p-1)/p} - 1} < \varepsilon.
\]

Inclusion (2.5.3) follows because \( h_{(a,b)}(c,z) < \varepsilon \) for all \( z \in \overline{B}(0,R^{1/p}) \), while (2.5.4) holds because \( h_{(c,d)}(a,z) < \varepsilon \) for all \( z \in R^2 \setminus B(0,R) \).

The Möbius transformation \( \mu(z) = R/z \) carries \( \overline{R^2} \setminus B(0,R^{(p-1)/p}) \) into \( \overline{B}(0,R^{1/p}) \) and \( \overline{B}(0,1) \) into \( \overline{R^2} \setminus B(0,R) \), while the points \( a^*,b^*,c^*, \) and \( d^* \) go to some points \( a, b, c, \) and \( d \) satisfying the conditions of the first part of the proposition. Using the Möbius invariance of (2.1), we obtain (2.5.5)–(2.5.7) from (2.5.2)–(2.5.4). \( \square \)

Two paths \( \gamma_1, \gamma_2 : [0,1] \to M \subset \overline{R^2} \) with the same endpoints \( a = \gamma_1(0) = \gamma_2(0) \) and \( b = \gamma_1(1) = \gamma_2(1) \) are called homotopic in \( M \) if there exists a continuous mapping \( (a \text{ homotopy}) \) \( F : [0,1] \times [0,1] \to M \) such that \( F(0,s) \equiv a, F(1,s) \equiv b, F(t,0) \equiv \gamma_1(t), \) and \( F(t,1) \equiv \gamma_2(t) \). For two Jordan arcs \( \gamma_1 \) and \( \gamma_2 \) in this situation we say that \( \gamma_1 \) contracts \( \gamma_2 \) in \( M \).

Note some topological properties of Jordan domains.

2.6.1. If \( D \subset \overline{R^2} \) is a Jordan domain then two arbitrary paths \( \gamma_1, \gamma_2 \subset \overline{D} \) with the same endpoints \( a,b \in D \) are homotopic in \( D \).