RATIONAL APPROXIMANTS FOR THE EULER CONSTANT
AND RECURRENCE RELATIONS

Four-Term Recurrence Relations for $\gamma$-Forms

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1. INTRODUCTION

Let $Q_n(x)$ be the polynomials defined by the generalized Rodrigues formula

$$Q_n(x) = \frac{(1-x)^{-1}e^x}{(n!)^2} \frac{d^n}{dx^n} x^n \frac{d^n}{dx^n} (1-x)^{2n+1}x^n e^{-x}.$$  (1)

Consider two mathematical constants, the Euler constant

$$\gamma := -\int_0^{\infty} \ln x e^{-x} dx$$

and the value of the integral exponential $e^{\text{Ei}(1, 1)}$, where

$$\text{Ei}(n, x) = \int_1^{\infty} \frac{e^{-xt}}{t^n} dt.$$  

Let $f_n$ and $g_n$ be the sequences of $\mathbb{Z}$-forms with respect to the constants $\gamma$ and $e^{\text{Ei}(1, 1)}$ generated by the polynomials $Q_n$ according to the formulas

$$f_n := p_n - \gamma q_n := \int_0^{\infty} Q_n(x) \ln x e^{-x} dx$$  (2)

and

$$g_n := e^{\text{Ei}(1, 1)} q_n - r_n := e \int_1^{\infty} Q_n(x) \ln x e^{-x} dx.$$  (3)

This paper is devoted to the proof of the following theorem.

**Theorem 1.** The integer coefficients $p_n$, $q_n$, and $r_n$ in forms (2) and (3) satisfy the recurrence relation

$$(16n - 15)q_{n+1} = (128n^2 + 40n^2 - 82n - 45)q_n - n^2(256n^3 - 240n^2 + 64n - 7)q_{n-1} + n^2(n - 1)^2(16n + 1)q_{n-2}$$

with initial conditions

$$p_0 = 0, \quad p_1 = 2, \quad p_2 = 31,$$

$$q_0 = 1, \quad q_1 = 3, \quad q_2 = 50,$$

$$r_0 = 0, \quad r_1 = 1, \quad r_2 = 24.$$
2. A RELATIONSHIP BETWEEN $\gamma$-FORMS AND THE RECURRENCE RELATIONS FOR $Q_n(1)$

As is known (see [1]), the polynomials $Q_n^{(\alpha_1, \alpha_2)}$ with $\deg Q_n^{(\alpha_1, \alpha_2)} = 4n$, which are defined by the Rodrigues formula

$$Q_n^{(\alpha_1, \alpha_2)}(x) = \frac{1}{(n!)^2}(1 - x)^{-\alpha_2}e^x \frac{d^n}{dx^n} x^{n+\alpha_2-\alpha_1} \frac{d^n}{dx^n}(1 - x)^{2n+1}x^{n+\alpha_1}e^{-x},$$

(5)
satisfy the system of orthogonality relations

$$\begin{align*}
\int_0^1 Q_n^{(\alpha_1, \alpha_2)}(x)x^\nu w_1(x) \, dx &= 0, \\
\int_0^1 Q_n^{(\alpha_1, \alpha_2)}(x)x^\nu w_2(x) \, dx &= 0, \\
\int_1^\infty Q_n^{(\alpha_1, \alpha_2)}(x)x^\nu w_1(x) \, dx &= 0, \\
\int_1^\infty Q_n^{(\alpha_1, \alpha_2)}(x)x^\nu w_2(x) \, dx &= 0,
\end{align*}$$

(6)

where

$$w_1(x) := x^\alpha_1(1 - x)e^{-x}, \quad w_2(x) := x^\alpha_2(1 - x)e^{-x}.$$  

It is easy to show (by subtracting the orthogonality relations in (6) and dividing the result by a constant) that the orthogonality relations (6) remain valid for

$$w_1(x) := x^\alpha_1(1 - x)e^{-x}, \quad w_2(x) := \frac{x^\alpha_2 - x^\alpha_1}{\alpha_2 - \alpha_1}(1 - x)e^{-x}.$$  

Therefore, letting $\alpha_2 \to \alpha_1$ and, then, $\alpha_1 \to 0$, we see that the polynomials $Q_n := Q_n^{(0,0)}$ (see (1) and (5)) satisfy the orthogonality relations (6) with weight functions

$$w_1(x) := (1 - x)e^{-x}, \quad w_2(x) := (1 - x)\ln e^{-x}.$$  

(7)

Consider the following functions of the second kind, which are related to the orthogonality system (6), (7):

$$R_n(z) = \int_0^\infty \frac{Q_n(x)}{z - x}(1 - x)\ln e^{-x} \, dx, \quad S_n(z) = \int_0^1 \frac{Q_n(x)}{z - x}(1 - x)\ln e^{-x} \, dx.$$  

It is known (see, e.g., [2]) that, by virtue of the orthogonality relations, the functions of the second kind generated by $Q_n(x)$ satisfy the same recurrence relations as the polynomials $Q_n(x)$, that is, the recurrence relations of Theorem 1 in [3] and, as a consequence, the recurrence relations of Theorems 1 and 2 in [4]. Thus, the forms

$$f_n = R_n(1), \quad g_n = eS_n(1)$$  

(see (2) and (3)) of interest to us satisfy the same recurrence relations as $Q_n(1)$, in particular, the eight-term recurrence relation of the corollary to Theorem 2 in [4].

3. DERIVATION OF A FOUR-TERM RECURRENCE RELATION FOR FORMS

The recurrence relations obtained in [4] made it possible to compute forms (2) and (3) up to $n$ amounting to several thousand, which, in turn, gave a rich material for experimentally studying rational approximants for the Euler constant. In particular, an analysis of common multipliers of the sequences

$$(ap_n + bp_{n-1} + cp_{n-2}) \quad \text{and} \quad (aq_n + bq_{n-1} + cq_{n-2})$$

(see (4) and (5)).