Ray Optics in the Field of a Nonminimal Dirac Monopole

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Abstract—On the basis of an analogy with the nonminimal SU(2)-symmetric Wu-Yang monopole with a regular metric, a solution describing a nonminimal U(1)-symmetric Dirac monopole is obtained. To take into account the curvature coupling of the gravitational and electromagnetic fields, we reconstruct the effective metrics of two types, the so-called associated and optical metrics. The optical metrics explicitly show that the curvature-induced birefringence effect takes place in the vicinity of a nonminimal Dirac monopole; these optical metrics are studied analytically and numerically.

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1. INTRODUCTION

The Dirac monopole as a specific static spherically symmetric solution to the minimal Einstein-Maxwell equations has become a subject of discussion in tens of papers, reviews and books (see, e.g., [1–7]). The motion of massive and massless particles, with and without an electric charge, is studied in detail (see, e.g., [8, 9] and references therein). In [10], we have introduced and discussed an SU(2)-symmetric Wu-Yang monopole of a new type, namely, a nonminimal monopole with a regular metric. Since the nonminimal Wu-Yang monopole is effectively Abelian, it is natural to consider the corresponding analog of this solution in the framework of nonminimal electrodynamics. Let us mention that nonminimal models with a magnetic charge have been discussed before (see, e.g., [11]), but the exact analytic regular solution obtained here as a direct reduction to the U(1) symmetry is new.

Another novelty of the present paper is an investigation of photon dynamics in the vicinity of the Dirac monopole taking into account the curvature coupling of the gravitational and electromagnetic fields. In the presence of a curvature-induced nonminimal interaction, the master equations for electromagnetic and gravitational fields in vacuum can be rewritten as a master equations in some effective anisotropic (quasi)medium [12, 13]. This means that two effective (optical) metrics can be introduced [14–16], so that the propagation of a photon in vacuum, interacting with the curvature, is equivalent to the photon motion in an effective space-time with the first or second optical metric, depending on the photon polarization. Even if real space-time has a regular metric, the optical metrics can be singular, admitting an interpretation in terms of the so-called “trapped surfaces” and “inaccessible zones” [17, 18]. We discuss this problem in Section 3. Numerical modelling of the photon orbits, presented in Section 4, supplements our conclusions.

2. MASTER EQUATIONS AND BACKGROUND FIELDS

We consider the Lagrangian

\[ L = \frac{R}{8\pi} + \frac{1}{2} F_{ik} F^{ik} + \frac{1}{2} R^{ikmn} F_{ik} F_{mn} \]  

(1)

to describe the background gravitational and magnetic fields in the framework of a nonminimal Einstein-Maxwell model with the nonminimal susceptibility tensor

\[ R^{ikmn} = \frac{q^2}{2} \left[ R \left( g^{in} g^{km} - g^{im} g^{kn} \right) \right. \]

\[ - 12 R^{ikmn} + 4 \left( R^{im} g^{kn} + R^{kn} g^{im} \right) \]

\[ - \left. R^{in} g^{km} - R^{km} g^{in} \right] \]

(2)

linear in the curvature. As usual, \( R^{ikmn} \) is the Riemann tensor, \( R^{mn} \) the Ricci tensor, \( R \) the Ricci scalar, and \( F_{mn} \) the Maxwell tensor. This nonminimal
susceptibility tensor can be obtained from the general one (see [13]) when the coupling constants \( q_1, q_2, q_3 \) are chosen as follows: \( q_1 = -q < 0 \), \( q_2 = 4q \), \( q_3 = -6q \). The ansatz for the space-time metric is

\[
\begin{align*}
\mathrm{ds}^2 &= N(r)dt^2 - \frac{dr^2}{N(r)} \\
&\quad - r^2 \left( \mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\varphi^2 \right),
\end{align*}
\]

(3)

Let us mention that the choice \( g_{tt}g_{rr} = -1 \) in our ansatz is supported by the result obtained in [10] for these relations between the coupling constants \( q_1, q_2, q_3 \). The equations of electrodynamics [14, 19]

\[
\nabla_k H^{ik} = 0, \quad \nabla_k F^{*ik} = 0
\]

(4)

with the constitutive equations

\[
H^{ik} = C^{ikmn} F_{mn}, \quad C^{ikmn} = \frac{1}{2} \left( g^{im} g^{kn} - g^{in} g^{km} \right) + R^{ikmn}
\]

(5)

are associated with the Lagrangian (1), where \( F^{*ik} \) is the dual Maxwell tensor and \( H^{ik} \) is the induction tensor. Eqs. (4) with (5) are satisfied identically when the potential of the static, spherically symmetric electromagnetic field outside a point-like magnetic charge \( \mu \) is of the form

\[
A_k = \delta^\varphi_k A_\varphi = -\delta^\varphi_k \mu (1 - \cos \theta).
\]

(6)
The corresponding field strength tensor has only one non-vanishing component

\[
F_{\theta \varphi} = -\mu \sin \theta,
\]

(7)

which does not depend on the nonminimal coupling parameter \( q \). Thus the well-known solution with a monopole-type magnetic field

\[
B^i \equiv F^{*ik} U_k = \delta^i_k \mu \sqrt{\frac{N}{r^2}}, \quad B(r) \equiv \sqrt{-B^2 B_i} = \frac{\mu}{r^2},
\]

(8)

satisfies the nonminimal Maxwell equations (4), (5), (2). The equations for the gravitational field

\[
R_{ik} - \frac{1}{2} R g_{ik} = 8\pi T_{ik}^{(\text{eff})},
\]

(9)

obtained by a direct variation procedure from the Lagrangian (1), are nonminimally extended since the effective stress-energy tensor \( T_{ik}^{(\text{eff})} \) has the form

\[
T_{ik}^{(\text{eff})} = T_{ik}^{(0)} + \frac{1}{4} g_{ik} F_{mn} F^{mn} - F_{im} F_k^m,
\]

(10)

The quantities \( T_{ik}^{(0)}, T_{ik}^{(1)}, T_{ik}^{(2)} \), and \( T_{ik}^{(3)} \) are given by

\[
T_{ik}^{(0)} = \frac{1}{4} g_{ik} F_{mn} F^{mn} - F_{im} F_k^m,
\]

(11)

\[
T_{ik}^{(1)} = R T_{ik}^{(0)} - \frac{1}{2} R_{ik} F_{mn} F^{mn}
\]

\[
- \frac{1}{2} g_{ik} \nabla^l \nabla_l (F_{mn} F^{mn}) + \frac{1}{2} \nabla^l \nabla_l (F_{mn} F^{mn}),
\]

(12)

\[
T_{ik}^{(2)} = -\frac{1}{2} g_{ik} \left[ \nabla_m \nabla_l (F_{mn} F^l) - R_{lm} F_{mn} F^n \right] - F_{ln} \left( R_{il} F_{kn} + R_{kl} F_{in} \right)
\]

\[
- R_{lm} F_{kn} F_{in} - \frac{1}{2} \nabla^l \nabla_l (F_{kn} F_k^n) + \frac{1}{2} \nabla_l \left[ \nabla_i (F_{kn} F^n) + \nabla_k (F_{in} F^{in}) \right],
\]

(13)

\[
T_{ik}^{(3)} = \frac{1}{4} g_{ik} R^{mnls} F_{mn} F_{ls}
\]

\[
- \frac{3}{4} F^{ls} \left( F_{in} R_{knls} + F_{kn} R_{inls} \right)
\]

\[
- \frac{1}{2} \nabla_m \nabla_l (F_{in} F_k^m + F_{kn} F_{in}^m).
\]

(14)

We are happy to stress that, with the ansatz (7) for the metric (3), the set of equations (9) transforms into a single equation:

\[
r N' \left( 1 + \frac{\kappa q}{r^2} \right) + N \left( 1 - 3q \frac{\kappa}{r^4} \right)
\]

\[
= 1 - \frac{\kappa}{2r^2} - 3q \frac{\kappa}{r^4},
\]

(15)

whose solution is

\[
N(r) = 1 + \frac{r^2 (\kappa - 4M r)}{2(r^4 + \kappa q)}.
\]

(16)

Here \( \kappa \) is a convenient constant, \( \kappa = 8\pi \mu^2 \). This solution is regular at the center (\( N(0) = 1 \)) and satisfies the asymptotic condition \( N(\infty) = 1 \). If the mass \( M \) is smaller than its critical value \( M_{\text{(crit)}} \), where

\[
M_{\text{(crit)}} = \frac{r_s}{6} \left( 4 + \frac{\kappa}{r_s^2} \right),
\]

\[
r_s = \frac{\sqrt{\kappa}}{2} \sqrt{\left( \sqrt{1 + \frac{48q}{\kappa}} + 1 \right)},
\]

(17)

the metric (3) with (16) has no horizons, as in [10].

3. ELECTRODYNAMIC DESCRIPTION OF PHOTON PROPAGATION

Linear electrodynamics allows us to consider the dynamics of test photons in terms of the microscopic field strength \( f_{ik} \) and the induction \( h_{ik} \) which satisfy the equations

\[
\nabla_k h^{ik} = 0, \quad h^{ik} = C^{ikmn} f_{mn},
\]

(18)

\[
\nabla_k f^{*ik} = 0.
\]