A Note on Proper Affine Vector Fields in Non-Static Plane Symmetric Space-Times

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Abstract—The most general form of non-static plane symmetric space-times is considered to study proper affine vector fields by using holonomy and decomposability, the rank of the $6 \times 6$ Riemann matrix and direct integration techniques. Studying proper affine vector fields in each nonstatic case of the above space-times it is shown that very special classes of the above space-times admit proper affine vector fields. We also discuss the Lie algebra in each case.

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1. INTRODUCTION

The most general form of non-static plane symmetric space-times is considered to study the existence of proper affine vector fields in non-static plane symmetric space-times by using holonomy and decomposability, the rank of the $6 \times 6$ Riemann matrix and direct integration techniques. Affine vector fields which preserve the geodesic structure and affine parameter of a space-time carry significant information and interest in Einstein’s theory of general relativity. It is therefore important to study this symmetry. Different approaches [2, 4, 5, 8–14] were adopted in studying affine vector fields. It is important to mention here that we will only consider nonstatic cases. The cases when the above space-time becomes static are studies in [2].

Let $(M, g)$ be a space-time with $M$ a smooth connected Hausdorff four-dimensional manifold and $g$ a smooth metric of Lorentz signature $(-, +, +, +)$ on $M$. The curvature tensor associated with $g$ through Levi-Civita connection $\Gamma$ is denoted in the component form by $R^a_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol $L$, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. The space-time $M$ will be assumed nonflat in the sense that the Riemann tensor does not vanish over any nonempty open subset of $M$.

A vector field $X$ on $M$ is called an affine vector field if it satisfies

$$X_{a;bc} = R_{abcd}X^d,$$  \hspace{1cm} (1)

where

$$R_{abcd} = g_{af}R^f_{bcd} = g_{af}(\Gamma^f_{bd,c} - \Gamma^f_{bc,d} + \Gamma^f_{ce}\Gamma^e_{bd} - \Gamma^f_{ed}\Gamma^e_{bc}).$$

If one decomposes $X_{a;b}$ on $M$ into its symmetric and skew-symmetric parts,

$$X_{a;b} = \frac{1}{2}H_{ab} + G_{ab},$$

then Eq. (1) is equivalent to

\begin{align*}
(i) & \quad H_{abc} = 0, \\
(ii) & \quad G_{abc} = R_{abcd}X^d, \\
(iii) & \quad G_{ab;c}X^c = 0.
\end{align*}

The proof of the above equation (1) implies (3), or equation (3) implies (1), as can be found in [3, 4]. If $H_{ab} = 2\epsilon_{gab}$, $c \in R$, then the vector field $X$ is called homothetic (and Killing if $c = 0$). The vector field $X$ is said to be proper affine if it is not homothetic vector field, and also $X$ is said to be a proper homothetic vector field if it is not a Killing vector field on $M$ [5]. Define the subspace $\mathcal{Z}_p$ of the tangent space $T_pM$ to $M$ at $p$ as those $k \in T_pM$ satisfying

$$R_{abcd}k^d = 0.$$  \hspace{1cm} (4)

The Lie algebra of a set of vector fields on a manifold is completely characterized by the structure constants $C^{a}_{bc}$ given in term of the Lie brackets by

$$[X_b, X_c] = C^{a}_{bc}X_a, \quad C^{a}_{bc} = -C^{a}_{cb},$$

where $X_a$ are the generators and $a, b, c = 1, 2, 3, \ldots, n$. In each case we discuss the Lie algebra for the affine vector fields.

2. AFFINE VECTOR FIELDS

Suppose that $M$ is a simple connected space-time. Then the holonomy group of $M$ is a connected
3. MAIN RESULTS

Consider a nonstatic plane-symmetric space-time in the usual coordinate system \((t, x, y, z)\) (labeled by \((x^0, x^1, x^2, x^3)\), respectively) with the line element \([7]\)

\[ds^2 = -e^{A(t,x)} dt^2 + e^{B(t,x)} dx^2 + e^{C(t,x)} (dy^2 + dz^2).\]  \hspace{1cm} (5)

The Ricci tensor Segre type of the above space-time is \(1,1(11)\) or \(2(11)\) or one of its degeneracies. The above space-time admits three linearly independent Killing vector fields which are

\[
\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}. \hspace{1cm} (6)
\]

The non-zero independent components of the Riemann tensor are

\[
R_{0101} = \frac{1}{4} [e^{A(t,x)} (A_x^2(t,x)) + 2 A_{xx}(t,x) - A_x(t,x) B_x(t,x) - e^{B(t,x)} (B_t^2(t,x) + 2 B_{tt}(t,x)) - A_t(t,x) B_t(t,x)) = \alpha_1,
\]

\[
R_{0202} = R_{0303} = -\frac{1}{4} e^{C(t,x)} - B(t,x) [e^{B(t,x)} (C_t^2(t,x) + 2 C_{tt}(t,x) - A_t(t,x) C_t(t,x)) - e^{A(t,x)} A_x(t,x) C_x(t,x))] = \alpha_2,
\]

\[
R_{1212} = R_{1313} = -\frac{1}{4} e^{C(t,x)} - A(t,x) [e^{A(t,x)} (C_x^2(t,x) + 2 C_{xx}(t,x) - B_x(t,x) C_x(t,x)) - e^{B(t,x)} B_t(t,x) C_t(t,x))] = \alpha_3,
\]

\[
R_{2323} = -\frac{1}{4} e^{A(t,x) + B(t,x) + 2 C(t,x)} [e^{A(t,x)} (C_x^2(t,x) - e^{B(t,x)} (C_t^2(t,x)))] = \alpha_4,
\]

\[
R_{0212} = R_{0313} = \frac{1}{4} e^{C(t,x)} [C_x(t,x) C_t(t,x) + 2 C_{tx}(t,x) - A_x(t,x) C_t(t,x) - B_t(t,x) C_x(t,x)) = \alpha_5.
\]

Writing the curvature tensor with components \(R_{abcd}\) at \(p\) as a \(6 \times 6\) symmetric matrix,

\[
R_{abcd} = \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_2 & 0 & \alpha_5 & 0 & 0 \\
0 & 0 & \alpha_2 & 0 & \alpha_5 & 0 \\
0 & 0 & 0 & \alpha_5 & 0 & \alpha_3 \\
0 & 0 & 0 & \alpha_5 & 0 & \alpha_3 \\
0 & 0 & 0 & 0 & 0 & \alpha_4
\end{pmatrix}.
\hspace{1cm} (7)
\]

As mentioned in section 2, the space-times which can admit proper affine vector fields have the holonomy type \(R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11}\) or \(R_{13}\), and the rank of the \(6 \times 6\) Riemann matrix is at most three. Therefore we are only interested in those cases when the rank of the \(6 \times 6\) Riemann matrix is less than or equal to three.