On the Bottom Summation

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Abstract—We consider summation of consecutive values $\varphi(v), \varphi(v+1), \ldots, \varphi(w)$ of a meromorphic function $\varphi(z)$, where $v, w \in \mathbb{Z}$. We assume that $\varphi(z)$ satisfies a linear difference equation $L(y) = 0$ with polynomial coefficients, and that a summing operator for $L$ exists (such an operator can be found—if it exists—by the Accurate Summation algorithm, or, alternatively, by Gosper’s algorithm when $\text{ord}L = 1$). The notion of bottom summation which covers the case where $\varphi(z)$ has poles in $\mathbb{Z}$ is introduced.

1. INTRODUCTION

The object of this note is to present the results of [1] related to the so-called “bottom summation” in a simpler form. The object of our investigation is correctness of the discrete Newton–Leibniz formula for definite summation in the case where a summing operator has been successfully constructed by the Accurate Summation algorithm [2] or by Gosper’s algorithm [3]. In the detailed proofs given in [1], many abstract notions were used. In addition, it was necessary to prove a number of auxiliary statements. As a result, the paper [1] is quite difficult to read. However, the main results of [1] are of some practical interest for computer algebra and their short presentation without complicated proofs can be useful.

Below, we present the main results of [1] in a simpler form and give some illustrations. Full proofs can be found in [1].

2. SUMMING OPERATORS

Let $E$ be the shift operator such that $E(f(k)) = f(k+1)$ for sequences $f(k)$, where $k \in \mathbb{Z}$, and $E(\varphi(z)) = \varphi(z+1)$ for analytic functions, $z \in \mathbb{C}$. Let

$$L = a_d(k)E^d + \ldots + a_1(k)E + a_0(k) \in \mathbb{C}(k)[E].$$

We say that an operator $R \in \mathbb{C}(k)[E]$ is a summing operator for $L$ if

$$(E-1) \circ R = 1 + M \circ L \quad (1)$$

for some $M \in \mathbb{C}(k)[E]$. We can assume without loss of generality that $\text{ord}R = \text{ord}L - 1 = d - 1$:

$$R = r_{d-1}(k)E^{d-1} + \ldots + r_1(k)E + r_0(k) \in \mathbb{C}(k)[E].$$

3. THE DISCRETE NEWTON–LEIBNIZ FORMULA

If a summing operator exists, then it can be constructed by the Accurate Summation algorithm [4] or, when $d = 1$, by Gosper’s algorithm [3]. At first glance, in those cases where $R \in \mathbb{C}(k)[E]$ exists, equality (1) gives us an opportunity to use the discrete Newton–Leibniz formula (DNLF)

$$\sum_{k = v}^{w-1} f(k) = g(w) - g(v)$$

for all integers $v < w$ and for any sequence $f$ such that $L(f) = 0$ taking $g = R(f)$. Indeed, we can apply both sides of $(E-1) \circ R = 1 + M \circ L$ to $f$. This gives

$$(E-1)(R(f)) = f + M(L(f)).$$

Set $g = R(f)$. Taking into account that $L(f) = 0$, we get

$$(E-1)g = f,$$

or, equivalently,

$$g(k+1) - g(k) = f(k).$$

As a consequence, the DNLF is applicable:

$$\sum_{k = v}^{w-1} f(k) = \sum_{k = v}^{w-1} (g(k+1) - g(k))$$

$$= g(w) - g(w-1) + g(w-1) - g(w-2) + \ldots + g(v+1) - g(v)$$

$$= g(w) - g(v)$$

(the telescoping effect).

However, it was shown that, if $R$ has rational-function coefficients that have poles in $\mathbb{Z}$, then this formula may give incorrect results (an example will be demon-
Example 1. Consider the sequence

\[ f(k) = \frac{2k - 3}{4^k}, \]

which satisfies the first-order recurrence relation \( 2(k + 1)(k - 2)f(k + 1) - (2k - 1)f(k) = 0. \)

Although Gosper’s algorithm succeeds on this sequence, producing \( R(k) = \frac{2k(k + 1)}{k - 2} \), and \( f(k) \) is defined for all \( k \in \mathbb{Z} \), the discrete Newton–Leibniz formula

\[
\sum_{k=0}^{w-1} f(k) = R(w) f(w) - R(0) f(0)
\]

\[
= \frac{2w(w + 1)(2w - 3)}{(w - 2)4^w}
\]

is not correct: if we assume that the value of \( \frac{2k - 3}{k} \) is 1 when \( k = 0 \) and -1 when \( k = 1 \) (as is common practice in combinatorics), then the expression on the right gives the true value of the sum only at \( w = 1 \).

4. THE BOTTOM SUMMATION

Suppose that \( L \) acts on analytic functions:

\[ L = a_d(z)E^d + \ldots + a_1(z)E + a_0(z) \in \mathbb{C}(z)[E]. \]  

We consider the summing operator (if it exists) in the form

\[ R = r_d(z)E^{d-1} + \ldots + r_1(z)E + r_0(z) \in \mathbb{C}(z)[E]. \]

Let \( \varphi(z) \) be a meromorphic solution of \( L(y) = 0 \).

It turns out that, if \( \varphi(z) \) has no pole in \( \mathbb{Z} \), then neither does \( R(\varphi(z)) \), and we can use the DNLF to sum values \( \varphi(k) \) for \( k = \nu, \nu + 1, \ldots, w \). So, such undesirable phenomena as demonstrated in Example 1 cannot occur if the elements of the sequence under summation are the values \( \varphi(k), k \in \mathbb{Z} \), of an analytic function \( \varphi(z) \), which satisfies (in the complex plane \( \mathbb{C} \)) the same difference equation with polynomial coefficients as does the original sequence (at integer points).

This follows from a stronger statement. The fact is that, even if \( \varphi(z) \) has some poles in \( \mathbb{Z} \), the summation task can nevertheless be performed correctly.

For any \( k \in \mathbb{Z} \), the function \( \varphi(z) \) can be represented by Laurent’s series

\[ \varphi(z) = c_{k, \rho_k}(z - k)^{\rho_k} + c_{k, \rho_k + 1}(z - k)^{\rho_k + 1} + \ldots \]

with \( \rho_k \in \mathbb{Z} \) and \( c_{k, \rho_k} \neq 0 \). If \( L(\varphi) = 0 \), then there exists the minimal element \( \rho \) in the set of all \( \rho_k, k \in \mathbb{Z} \). This \( \rho \) we call the depth of \( \varphi(z) \) and denote it by \( \text{depth}(\varphi) \).

We associate with \( \varphi(z) \) the sequence \( f(k) \) such that \( f(k) = c_{k, \rho_k} \) if \( \rho_k = \rho \), and \( f(k) = 0 \) otherwise. This \( f(k) \) we call the bottom of \( \varphi(z) \) and denote it as \( \text{bott}(\varphi) \).

We illustrate these notions by the following simple example.

It is well known that \( \Gamma(z) \) has finite values when \( z = 1, 2, \ldots \) and has simple poles when \( z = 0, -1, -2, \ldots \). We have

\[ \text{depth}(\Gamma) = -1 \]

and

\[ \text{bott}(\Gamma)(k) = \begin{cases} 0, & \text{if } k > 0 \\ (1 - k)^{k+1}, & \text{if } k \leq 0. \end{cases} \]

If we consider \( \Gamma(z) \) only in the half-plane \( \text{Re}z > 0 \), then its depth is 0 and the bottom is the sequence

\[ f(k) = (k - 1)! , \quad k = 1, 2, \ldots. \]

Proposition 1. Let \( L(\varphi) = 0 \). Then, \( L(\text{bott}(\varphi)) = 0 \).

Proposition 2. Let \( L(\varphi) = 0 \), and let \( R \) be a summing operator for \( L \). Then, \( \text{depth}(\varphi) = \text{depth}(R(\varphi)) \).

Theorem 1 (on the bottom summation). Let \( L(\varphi(z)) = 0 \), and let \( R \) be a summing operator for \( L \). Denote \( \psi(z) = R(\varphi(z)) \). Then the bottom summation formula

\[
\sum_{k=\nu}^{w-1} \text{bott}(\varphi)(k) = \text{bott}(\psi)(w) - \text{bott}(\psi)(\nu)
\]

is valid for any \( \nu < w \). In particular, if \( \varphi \) has no pole in \( \mathbb{Z} \) (i.e., \( \text{depth}(\varphi) \geq 0 \)), then the function \( \psi(z) \) has no pole in \( \mathbb{Z} \), and the discrete Newton–Leibniz formula

\[
\sum_{k=\nu}^{w-1} \varphi(k) = \psi(w) - \psi(\nu)
\]

is valid for any \( \nu < w \).

Example 1 (continued). Assume that the value of

\[ \frac{2k - 3}{k} \]

is defined as

\[
\lim_{z \to k} \Gamma(2z - 2) \Gamma(z - 2)
\]

(3)

this is a natural extension of the formula