1. INTRODUCTION

In this paper, we use the theory of dynamical systems and numerical analysis to show the existence of stationary and traveling fronts in the generalized one-dimensional Swift–Hohenberg equation (see [1–5]). Additionally, we examine the stability properties of the resulting fronts. This work continues the study of this equation begun in [3, 4], where pulse-type solutions were investigated.

The generalized Swift–Hohenberg (GSH) equation

\[ u_t = \alpha u + \beta u^2 - u^3 - (1 + \partial^2/\partial x^2)^2 u \]  

was derived with \( \beta = 0 \) by Swift and Hohenberg in [1]. It models the appearance of one-dimensional periodic patterns of the form \( u = u(x), u(x + 2\pi/k) = u(x) \) when \( \alpha \) becomes positive. These periodic patterns were found to be stable only within a certain range of wave numbers \( k \). Temporal instability beyond this range is known as Eckhaus instability (see [6, 7]). The quadratic term \( \beta u^2 \) was first introduced into the GSH equation phenomenologically in [8] in order to model the threshold character of periodic pattern formation. Later, the resulting equation was found to be a useful model for the study of phenomena in various scientific areas from fluid dynamics and chemistry [1, 2, 7, 9] to nonlinear optics [10, 11].

Our interest in this paper focuses primarily on the study of localized solutions to the GSH equation, i.e., solutions that are stabilized as \( |x| \to \infty \). As is frequently done, we distinguish two types of such solutions. A stationary (i.e., independent of \( t \)) localized solution is called a stationary pulse if the limits as \( x \to \pm \infty \) exist and are identical and is called a stationary front if these limits are different. The physical interpretation of such solutions and their significance for the theory of dissipative structures have been discussed in numerous publications (see, e.g., [12]).

We try to mathematically substantiate the phenomena associated with the existence and origin of such structures, since, in our view, this would be useful for describing and searching for similar structures in other physically interesting equations. Based on our results, we develop numerical procedures for searching for and extending such solutions. The study relies heavily on the Hamiltonian structure of a system of first-order differential equations derived from the stationary equation. The Hamiltonian property of the system helps us to study it, but, strangely enough, this property, though noted previously, has been rarely used in studies related to the GSH equation. Another useful property of this system is that it is reversible with respect to...
some involution of the phase space, which follows from the evenness of the derivatives involved in the equation. All these points will be discussed in more detail below.

An existence theorem for solutions to Eq. (1) is derived by the standard argument and can be found, for example, in [4]. On an interval with periodic or other suitable boundary conditions, this equation is a gradient one; i.e., it can be written as

$$u_t = -\frac{\delta F}{\delta u},$$

where $F(u)$ is a functional. Therefore, its solutions $u(t, x)$ tend to stationary solutions $u(x)$ as $t \to \infty$. When considered on the entire line $\mathbb{R}$, the equation is no longer gradient, but it has traveling-front solutions $u(x - ct)$, more complex traveling-front solutions $u(x, x - ct)$ [13], and coarsening phenomena are possible [14].

2. STATIONARY SOLUTIONS

First, we recall some results concerning the stationary solutions of the GSH equation [3, 4]. Its stationary solutions (i.e., $u = u(x)$) satisfy the equation

$$u'''' + 2u'' - (\alpha - 1)u - \beta u^2 + u^3 = 0. \quad (2)$$

This equation has the type of the highest Lagrange–Euler–Ostrogradsky equation [15]. Therefore, introducing the new variables

$$u = q_1, \quad u' = q_2, \quad -(u' + u'') = p_1, \quad u + u'' = p_2, \quad (3)$$

we can reduce it to a Hamiltonian system with two degrees of freedom with “time” $x$

$$q_1' = q_2 = H_{p_1}, \quad p_1' = p_2 - \alpha q_1 - \beta q_1^2 + q_1^3 = -H_{q_1}, \quad (4)$$

$$q_2' = p_2 - q_1 = H_{p_2}, \quad p_2' = -p_1 = -H_{q_2},$$

and with the Hamiltonian

$$H = p_1q_2 - p_2q_1 + \frac{p_2^2}{2} + \frac{\alpha q_1^2}{2} + \frac{\beta q_1^4}{3} - \frac{q_1^4}{4}.$$ 

This system is also reversible with respect to the reversible involution $\sigma_1$: $(q_1, q_2, p_1, p_2) \mapsto (q_1, -q_2, -p_1, p_2)$. Recall that reversibility (see, e.g., [16]) means that, if the system has a solution $X(x) = (q_1(x), q_2(x), p_1(x), p_2(x))$, then $Y(x) = \sigma_1(X(-x)) = (q_1(-x), -q_2(-x), -p_1(-x), p_2(-x))$ is a solution to the system as well. In what follows, we assume without loss of generality that $\beta$ is nonnegative.

For all parameter values, system (4) has the equilibrium $O = (0, 0, 0, 0)$, which corresponds to the spatially homogeneous reference solution $u \equiv 0$ to Eq. (1). Homoclinic orbits to this equilibrium, i.e., solutions to the system that tend to the equilibrium as $x \to \pm\infty$, correspond to stationary pulses of Eq. (2).

The type of the equilibrium $O$ depends only on $\alpha$. Specifically, $O$ is a saddle–focus if $\alpha < 0$ (which means that the matrix of the linearized system has four complex conjugate eigenvalues $\pm \sigma \pm i\rho$, $\sigma \neq 0$, $\rho \in \mathbb{R}$) and is an elliptic point if $0 < \alpha < 1$ (the matrix of the linearized system has two different pairs of purely imaginary eigenvalues). If $\alpha > 1$, the point $O$ is a saddle–center; i.e., it has two nonzero real and two purely imaginary eigenvalues. In the case of a saddle–focus, in the neighborhood of the equilibrium, there are two local smooth two-dimensional manifolds $W^s$ and $W^u$ through $O$ that contain orbits tending to $O$ as $x \to -\infty$ and $x \to +\infty$, respectively. The 3-dimensional level $H = H(O) = 0$ contains these 2-dimensional manifolds ($H$ is an integral of the system), and, when extended along orbits of the system, they can intersect transversally along some orbits that are homoclinic to $O$. In the case of transversal intersection, these orbits persist under small variations in the parameters of the system, since the equilibrium and its stable and unstable manifolds depend smoothly on variations in the parameters if the Hamiltonian depends smoothly on these parameters (see [17]). Thus, the most natural domain for searching for homoclinic and heteroclinic orbits associated with $O$ is the half-plane $\alpha < 0$. It is this domain where the equation is studied below.

System (4) has other equilibria for parameter values at which $d = \beta^2 + 4(\alpha - 1)$ is positive. Specifically, two equilibria $P_1 = (u_1, 0, 0, u_1)$ and $P_2 = (u_2, 0, 0, u_2)$ are created after crossing the curve $d = 0$ in the parameter plane; here, $u_1 = (\beta - \sqrt{d})/2 < (\beta + \sqrt{d})/2 = u_2$. For $\beta \geq 0$, the equilibrium $P_2$ is an elliptic point for $(\alpha, \beta)$ between two asymptotically convergent (as $\alpha \to -\infty$) curves $d = 0$ and $L: \beta = (3 - 2\alpha)/\sqrt{2 - \alpha}$, $\alpha \leq 3/2$,