Let \( M_n(\mathbb{C}) \) be the space of complex \( n \times n \) matrices. Takagi’s decomposition of a symmetric matrix \( A \in M_n(\mathbb{C}) \) is defined by the following classical theorem (see [1, Chapter 4, Corollary 4.4.4]).

**Theorem 1** (Takagi). Let \( A = A^t \). Then there exist a unitary matrix \( U \) and a nonnegative diagonal matrix \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) \) such that

\[
A = U \Sigma U^T.
\] (1)

Decomposition (1) can be interpreted in two different ways. On the one hand, this is an analog for complex symmetric matrices of the eigenvalue decomposition of Hermitian matrices. On the other hand, equality (1) is a singular value decomposition of \( A \) that takes into account the symmetry of this matrix. Moreover, \( \sigma_1, \ldots, \sigma_n \) are the singular values of \( A \).

If \( n \geq 5 \), then Takagi’s decomposition of a general symmetric matrix \( A \in M_n(\mathbb{C}) \) cannot be obtained by performing a finite number of arithmetic operations and using (a finite number of) root extractions. Otherwise, the eigenvalue problem of complex symmetric matrices would be solvable by radicals. Since every matrix \( A \in M_n(\mathbb{C}) \) is similar to some symmetric matrix, this would imply that the eigenvalue problem is solvable by radicals for all complex matrices. It is well known that this is not true.

Our goal in this short paper is to indicate a subclass of symmetric matrices for which Takagi’s decomposition can be realized by a finite sequence of arithmetic operations and quadratic radicals. This subclass consists of matrices that are simultaneously symmetric and unitary.

Thus, one can say that the problem of calculating Takagi’s decomposition of a unitary symmetric matrix is solvable by quadratic radicals. We prove this fact in Sections 2 and 3. In Section 4, an analog of this fact concerning a certain subclass of Hermitian matrices is justified.

Let \( A \in M_n(\mathbb{C}) \) be a unitary symmetric matrix. All the singular values of a unitary (and not necessarily symmetric) matrix are unities. Consequently, Takagi’s decomposition of \( A \) takes the form

\[
A = U U^T.
\] (2)

Thus, the problem is reduced to finding a unitary matrix \( U \) (which need not be symmetric).

Fix an arbitrary nonzero vector \( x \in \mathbb{C}^n \). By calculating its 2-norm and by normalizing \( x \), we can assume that \( x \) is a unit vector. There are two possibilities for this vector:

**Case (a).** The vectors \( A^k x \) and \( x \) are collinear.

**Case (b).** The vectors \( A^k x \) and \( x \) are linearly independent.

---

1 The article was translated by the author.
In Case (a), there exists a scalar \( \mu \) such that
\[
A\hat{x} = \mu x. \tag{3}
\]
Since \( A \) is a unitary matrix, we have
\[
\|A\hat{x}\|_2 = \|\hat{x}\|_2 = 1 = \|\mu x\|_2 = |\mu|;
\]
that is,
\[
|\mu| = 1.
\]
It turns out that, by modifying \( x \), we can reduce Case (b) to an equality of type (3). Indeed, define
\[
y = A\hat{x} + x. \tag{4}
\]
By assumption, \( y \neq 0 \). Now we obtain
\[
A\hat{y} = A(\hat{A}x + x) = A\hat{A}x + A\hat{x} = AA^*x + A\hat{x} = x + A\hat{x} = y.
\]
Here, the symmetry of \( A \) was used in the third equality, and its unitarity was used in the penultimate equality. Therefore,
\[
A\hat{y} = y. \tag{5}
\]
To proceed, we need to normalize \( y \) in the same way as \( x \) was earlier normalized. Note that relation (3) can also be reduced to form (5). Indeed, let \( \nu \) be a square root of \( \mu \). Then (3) can be rewritten as
\[
A(\nu x) = \nu x.
\]
It remains to set \( y = \nu x \).

Thus, in every case, a unit vector \( y \) satisfying equality (5) can be found by a finite computation involving arithmetic operations and quadratic radicals.

3. Let \( V_1 \in M_n(\mathbb{C}) \) be a unitary matrix that has the vector \( y \) found in Section 2 as its first column. Such a matrix exists because every unit vector can be complemented to an orthonormal basis of the unitary space \( \mathbb{C}^n \).

In view of (5), the matrix \( B_1 = AV_1 \) has the form
\[
B_1 = (yb_2 \ldots b_n).
\]
Define
\[
A_1 = V_1^* B_1 = V_1^* AV_1
\]
and use the fact that \( y \) is orthogonal to the other columns of \( V_1 \). This implies that the subdiagonal entries in the first column of \( A_1 \) are zero, while the diagonal entry, which is equal to the 2-norm of \( y \), is unity. Since \( A_1 \) is obtained by a unitary congruence transformation of \( A \), it remains a symmetric (and, certainly, a unitary) matrix. Consequently, all the off-diagonal entries in the first row of \( A_1 \) are also zero. Thus, \( A_1 \) is a matrix of the form
\[
A_1 = 1 \oplus \hat{A},
\]
where \( \hat{A} \) is a symmetric unitary matrix of order \( n - 1 \). The calculation of \( A_1 \) completes the first step in the algorithm for constructing Takagi’s decomposition.

At the second step of the algorithm, similar operations are performed with the matrix \( \hat{A} \). This results in a unitary matrix \( \hat{V}_2 \) of order \( n - 1 \) such that \( \hat{V}_2 A \hat{V}_2 \) is a matrix of the form
\[
1 \oplus \hat{A},
\]
where \( \hat{A} \) is a symmetric unitary matrix of order \( n - 2 \). Setting
\[
V_2 = 1 \oplus \hat{V}_2, \quad A_2 = V_2^* A_1 V_2,
\]
\[3.\]