Optimal Monotonization of a High-Order Accurate Bicompact Scheme for the Nonstationary Multidimensional Transport Equation

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Abstract—A hybrid scheme is proposed for solving the nonstationary inhomogeneous transport equation. The hybridization procedure is based on two baseline schemes: (1) a bicompact one that is fourth-order accurate in all space variables and third-order accurate in time and (2) a monotone first-order accurate scheme from the family of short characteristic methods with interpolation over illuminated faces. It is shown that the first-order accurate scheme has minimal dissipation, so it is called optimal. The solution of the hybrid scheme depends locally on the solutions of the baseline schemes at each node of the space-time grid. A monotonization procedure is constructed continuously and uniformly in all mesh cells so as to keep fourth-order accuracy in space and third-order accuracy in time in domains where the solution is smooth, while maintaining a high level of accuracy in domains of discontinuous solution. Due to its logical simplicity and uniformity, the algorithm is well suited for supercomputer simulation.

Keywords: transport equation, bicompact schemes, short characteristic method, monotone schemes, minimal dissipation, hybrid schemes.

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1. INTRODUCTION

A large class of problems in science and engineering requires the numerical solution of the radiative transfer equation or the transport equation for uncharged particles. The linear transport equation is one of the simplest partial differential equations. The general form of this equation for the distribution function is

$$\frac{1}{\nu} \frac{\partial u}{\partial t} + \Omega \nabla u + \sigma u = Q,$$

where $u(t, r, \Omega, E)$ is the distribution function in the phase space; $\nu$ is the velocity of the group of particles with energy $E$; $\Omega$ is the direction of motion of this group of particles; $\sigma(r, t, E)$ is the absorption coefficient; and $Q(u, t, r, \Omega, E)$ is a source that generally includes a linear scattering integral, a plasma self-emission source, and fission terms for reactor problems \cite{1, 2}, which themselves depend on the distribution function. Thus, in the general case, (1) is an integrodifferential equation, which is usually solved by applying the method of source iteration. According to this method, the right-hand side is calculated using the distribution function from the preceding iteration step and is considered known in the computation of a new approximation to the solution. In the construction of a scheme for the differential operator on the left-hand side of the equation, the structure of its right-hand side is of no interest.

In radiative gas dynamics, there are problems in which radiation can be assumed to be stationary (radiative transfer times are much less than gasdynamic times) \cite{3, 4} and problems with an essentially nonstationary interaction of radiation with substance. An example of the latter problems is the interaction of intense laser beams with targets (see \cite{5}). In this case, the nonstationary transport equation has to be com-
puted. Allowance for nonstationarity is important in reactor problems, especially in groups of thermal neutrons, whose speeds are low as compared to those of fission neutrons. In this paper, we consider the following nonstationary transport equation in Cartesian coordinates:

$$\frac{1}{v} \frac{\partial u}{\partial t} + \Omega_x \frac{\partial u}{\partial x} + \Omega_y \frac{\partial u}{\partial y} + \Omega_z \frac{\partial u}{\partial z} + \sigma u = Q. \quad (2)$$

Boundary conditions for Eq. (2) in a convex domain are set for incoming directions. This means that the values of $u$ are considered known in flight directions $\Omega$ that make an obtuse angle with the outward normal $n$ to the computational domain $G$:

$$u(t, \mathbf{r}, \Omega, E)|_{\partial G} = \phi(t, \mathbf{r}, \Omega, E), \quad (n, \Omega) < 0. \quad (3)$$

There are numerous publications concerning methods for solving the transport equation (see, e.g., [6]). These methods can be formally divided into two classes: those extendable to nonlinear hyperbolic systems of equations, for example, to gas dynamics equations and those applied to the inhomogeneous transport equation (2) in physical applications. Either class has its own features imposing constraints on the properties of the schemes involved. A fourth-order accurate bicompact scheme for the nonstationary transport equation and systems of hyperbolic equations was proposed and examined in [7–10]. A bicompact scheme is constructed on the minimum two-point stencil in each direction on orthogonal structured grids. A scheme with an order of accuracy higher than the second can be constructed using differential consequences of Eq. (2) and closing the system of equations with the help of the Euler–Maclaurin formula.

A compact approximation within a single mesh cell is convenient for problems with contact discontinuities. Moreover, with such an approximation, problems can be computed on highly nonuniform meshes with the preservation of the order of convergence. The first ideas of constructing compact schemes of higher order accuracy appeared, for example, in [11, 12], where such schemes were constructed on three-point stencils in each spatial direction.

A fourth-order scheme derived as a stationary limit of a nonstationary scheme was presented and studied in [13]. However, according to Godunov’s theorem [14], schemes with an order of accuracy higher than the first are nonmonotone and generate spurious oscillations (Gibbs phenomenon) in the numerical solution within high-gradient regions. The construction of a monotone scheme with minimal dissipation for the stationary transport equation in two dimensions was addressed in [15].

In practical problems, we are interested not only in the convergence of the numerical solution under mesh refinement, but most frequently in its correct qualitative behavior on a given grid; moreover, an important requirement is that the scheme be monotone or, at least, positive. This is necessary primarily for stable computation of the scattering integrals appearing on the right-hand side of the transport equations, as well as for correct computation of linear-fractional functionals of the distribution function that determine the quasi-diffusion coefficients and the coefficients in the boundary conditions when the quasi-diffusion method is used for convergence acceleration of scattering and fission iterations [16, 17]. Thus, the monotonicity of the scheme is frequently a crucial requirement for the possibility of obtaining a numerical solution of the problem. Moreover, it is naturally desirable that, in domains where the solution is smooth, its numerical counterpart be of high order of accuracy. Accordingly, the problem arises of finding an optimal monotonizer for the difference scheme used. Numerous publications are available concerning the design of monotone schemes. A possible technique is that, in steep-gradient regions, a high-order accurate scheme is switched to a first-order accurate one as, for example, in Fedorenko’s scheme [18]. However, this stiff switch may cause the lack of convergence of source iterations. A more interesting idea is to construct a scheme in some space of schemes so that its final coefficients are determined by additional monotonicity conditions; such a scheme is referred to as quasi-monotone (see [19]).

In this paper, we construct a high-order accurate bicompact scheme that is monotone and/or positive. Without loss of generality, the basic points concerning the construction method and the properties of this scheme can be described as applied to the one-dimensional nonstationary transport equation

$$L_1(u) = 0, \quad L_1(u) = \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + \sigma u - Q \quad (4a)$$

and its two-dimensional counterpart

$$L_2(u) = 0, \quad L_2(u) = \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + \sigma u - Q, \quad (4b)$$

assuming that $a > 0$ and $b > 0$. If the equations remain dimensional in time and the problem is computed in physical units, then, in all subsequent formulas, the time step $\tau$ has to be replaced by $\nu \tau$. 

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