INTRODUCTION

Various types of localized waves are known to occur in nature. For example, Rayleigh waves propagating along the free surface of an elastic body (for quasi-Rayleigh waves, see [1]) or flexural edge waves [2] that are known from the theory of thin elastic plates. The present paper investigates waves that propagate along a periodic set of point inhomogeneities in an elastic plate. The cases of an isolated plate and a plate that is in contact with an acoustic medium are considered. Since a plate possesses rigidity, its oscillations are described by a fourth-order equation, for which the Green function is finite. Such problems allow one to consider point inhomogeneities, i.e., attached masses [6, 7]. In the case of other types of inhomogeneities (attached moments of inertia, cuts, holes, etc.), the conditions at a point cannot be formulated in classical form. In this case, it is necessary to use asymptotic considerations related to the concept of zero-radius potentials [3]. These considerations [4] underlie the generalized point models, which, in addition to the advantages of the classical point models, have a wider area of application. In particular, the generalized point models allow corrections due to the size of inhomogeneities and modeling the latter not only in the form of attached masses, but also in the form of holes.

The present study begins with considering a periodic set of bodies attached to a plate. It is assumed that the area of attachment is small compared to the characteristic wavelength of the wave process. This allows the assumptions that the attachments are of a point character and the action of a body on the plate is only due to the mass of the body. The inclusion of the moment of inertia in consideration is possible in terms of the generalized point models, which should be the subject of a subsequent paper. Below, the case of a periodic set of holes is considered by representing them in terms of the generalized point models [4]. In both the case of point masses and the case of holes, the problem is reduced to dispersion equations expressed in explicit quadrature form. The dispersion equations are studied numerically for different parameters of the plate and the acoustic medium.

It should be noted that waves propagating along a periodic set of bodies have been known in acoustics (see, e.g., [5]).

AN ISOLATED PLATE WITH A PERIODIC SET OF POINT MASSES

Let us consider an isolated thin elastic plate. Let identical masses \( M \) be attached to it at the points \( x = 0, y = jd \), where \( j = -\infty, \ldots, -1, 0, 1, \ldots, +\infty \). Then, the flexural displacement field in the plate with attached masses obeys a biharmonic equation satisfied everywhere except for the mass attachment points:

\[
\Delta^2 w - k_0^2 w = \sum_{j=-\infty}^{+\infty} c_j \delta(x) \delta(y - jd). \tag{1}
\]

Here, \( k_0 = (\omega^2 \rho h/D)^{1/4} \) is the wave number of flexural waves in the plate, where \( \omega \) is the frequency, \( \rho \) is the density of the plate, \( h \) is its thickness, and \( D \) is its bending stiffness. The amplitudes \( c_j \) of passive sources are determined from the contact conditions:

\[
c_j = \frac{\omega^2 M}{D} w(0, jd), \quad j = -\infty, \ldots, +\infty. \tag{2}
\]

Note that Eqs. (2) can be substituted in Eq. (1), as was done in, e.g., [6]. However, in a more difficult case of holes, it is necessary to deal with the separately formulated equation and the conditions imposed on the amplitudes of passive sources at the inhomogeneity positions.
The field excited in the plate by an incident wave $w^{(i)}$ is usually determined in two stages. First, the general solution to Eq. (1) is obtained. This solution can be represented in the form of an infinite sum of Green functions:

$$g_d(\rho, \rho_0) = \frac{i}{8k_0^2}D\left[H_0^{(1)}(k_0\rho - \rho_0) - H_0^{(1)}(ik_0\rho - \rho_0)]\right],$$  \hspace{1cm} (3)$$

$$w = \sum_{j=-\infty}^{+\infty} c_j g_d(x, y; 0, jd).$$  \hspace{1cm} (4)$$

Then, the total field $w + w^{(i)}$ is substituted in the contact conditions, which yields a system of equations in the amplitudes $c_j$.

If a plane wave

$$w^{(i)} = A\exp(ik_0x\cos \varphi_0 + ik_0y\sin \varphi_0),$$

is incident on the plate, a shift of the origin of coordinates along the $y$ axis only leads to the appearance of the factor $\exp(ik_0d\sin \varphi_0)$ in the expression for the field. Thus, we obtain

$$c_j = ce^{i\alpha j}, \quad \alpha = k_0d \sin \varphi_0.$$  \hspace{1cm} (5)$$

Here, $c = c_0$ is the amplitude of the passive source positioned at the point $x = y = 0$. Using Eqs. (5), it is possible to simplify general solution (4). For this purpose, we represent the Hankel functions, which appear in Eq. (3) for the Green function, in the form of Fourier integrals:

$$w = \sum_{j=-\infty}^{+\infty} ce^{i\alpha j} \int e^{i(y-jd)} \left(\frac{e^{-\sqrt{\mu^2+k_0^2}|x|}}{\sqrt{\mu^2 - k_0^2}} - \frac{e^{-\sqrt{\mu^2+k_0^2}|x|}}{\sqrt{\mu^2 + k_0^2}}\right) d\mu.$$ 

We divide the sum into three parts: for $j < 0$, $j = 0$, and $j > 0$. In the integrals with $j < 0$, we change the summation index $j \rightarrow -j$; in the integrals with $j > 0$, we change the integration variable $\mu \rightarrow -\mu$. Now, if we bend the ends of the integration contours to the upper half-plane, the integrals will be absolutely convergent and we can change the order of summation and integration. Performing the summation of geometric progressions, we obtain

$$w = \frac{ci}{8k_0^2} \left[H_0^{(1)}(k_0\sqrt{x^2 + y^2}) - H_0^{(1)}(ik_0\sqrt{x^2 + y^2})\right] + \frac{c}{8\pi k_0} \int \left(\frac{e^{\mu y}}{e^{-\mu d - m} - 1} + \frac{e^{-\mu y}}{e^{\mu d + m} - 1}\right) \times \left(\frac{e^{-\sqrt{\mu^2 - k_0^2}|x|}}{\sqrt{\mu^2 - k_0^2}} - \frac{e^{-\sqrt{\mu^2 + k_0^2}|x|}}{\sqrt{\mu^2 + k_0^2}}\right) d\mu,$$

which yields

$$w(0, 0) = \frac{ci}{8k_0^2} + \frac{c}{8\pi k_0} \int \Phi(\mu, \alpha) \left(\frac{1}{\sqrt{\mu^2 - k_0^2}} - \frac{1}{\sqrt{\mu^2 + k_0^2}}\right) d\mu, \hspace{1cm} (6)$$

$$\Phi(\mu, \alpha) = \frac{1}{e^{-\mu d - m} - 1} + \frac{1}{e^{\mu d + m} - 1}. \hspace{1cm}$$

The integration contour can be deformed so as to obtain loops that envelope the cuts of radicals with the branching points at $\mu = k_0$ and $\mu = ik_0$. Calculating the jumps and changing to the variables $\alpha$ according to the formulas $\mu = k_0(1 + it)$ and $\mu = ik_0(1 + t)$, respectively, we obtain

$$w(0, 0) = \frac{ci}{8k_0^2} + \frac{c}{2\pi k_0} \left[F(-i) - F(1)\right], \hspace{1cm}$$

$$F(\chi) = \frac{1}{\sqrt{1 + \alpha}}\int \left[1 + 2\chi e^{2k_0d + 2k_0d} - 2\cos(\alpha)e^{k_0d + 2k_0d} + 1\right] d\alpha.$$ 

Substituting the above expression in the contact conditions, we arrive at the expression for the amplitude,

$$c = \frac{D}{\omega^2 M} - \frac{A}{8k_0^2} \frac{\Delta(\alpha)}{2\pi k_0} \left[F(-i) - F(1)\right].$$  \hspace{1cm} (7)$$

Calculations show that, in Eq. (7), the denominator is nonzero for $|\alpha| \leq k_0d$, which means that the problem has a solution. However, if the incident field is absent ($A = 0$), the restriction $|\alpha| \leq k_0d$ is lifted and it is possible to find a value of the parameter $\alpha = \alpha^* > k_0d$ such that the denominator in Eq. (7) will be zero. This case corresponds to a wave propagating along the set of attached masses.

Let us analyze the dispersion equation

$$\Delta(\alpha) = \frac{2\pi}{m} - \frac{\pi}{4} - F(-i) + F(1) = 0$$  \hspace{1cm} (8)$$

for real values of $\alpha$. Here,

$$m = \frac{\omega^2 M}{k_0^2 D},$$

is the reduced mass. Since $\alpha$ only appears in the dispersion equation in the form of $\cos \alpha$, it is sufficient to consider Eq. (8) within the interval $0 \leq \alpha \leq \pi$.

The dispersion equation is a complex one. However, it can be shown that, for sufficiently small values of $k_0d$, there is an interval of the values of $\alpha$ within which Eq. (8) is real. The imaginary part of the dispersion equation is calculated in Appendix A:

$$\text{Im}\Delta(\alpha) = \frac{\pi}{2k_0d} \sum_{n} \frac{1}{\sqrt{1 - t_n^2}},$$  \hspace{1cm} (9)$$