Stagnant Waves in an Elastic Wedge-Shaped Plate

Kh. B. Tolipov
South Ural State University, pr. Lenina 76, Chelyabinsk, 454080 Russia
e-mail: thb@susu.ac.ru
Received July 30, 2012

Abstract—Stagnant antisymmetric waves in an elastic wedge-shaped plate were studied theoretically and experimentally. The solution to the nonuniform problem was obtained in the approximation of a small and smooth field nonuniformity. The vibration amplitudes in a standing wave were calculated and measured for a small wedge angle. Experimental measurements were performed on an original setup.

Keywords: nonuniform wave, interference, wedge-shaped plate, wave field, elasticity theory
DOI: 10.1134/S1063771013040167

INTRODUCTION

As a rule, an nonuniforms structure of acoustic waves is a consequence of acoustic anisotropy of the medium in which waves propagate. In particular, the inhomogeneous structure of a Rayleigh wave reflects the anisotropy of the near-surface region of the half-space [1]. It is also obvious that a wedge-shaped plate (WSP) that has two near-surface regions is also an acoustically anisotropic medium. It should be noted that acoustic waves in a WSP have been studied incomprehensively. The absence of a physicomathematical model of propagation of these waves resulted in studies restricted to either numerical calculations or empirical dependences [2–6], which have only qualitatively accounted for the acoustic phenomena in a WSP.

Investigations show that the propagation of a surface wave in a plate is accompanied by fundamentally new effects. For example, when a Rayleigh wave travels within a certain region adjacent to an edge, its loss of stability (stationarity) is observed. In this case, a special type of surface waves arises, which during its motion is accompanied by continuous reconstruction of the acoustic field and emission of spatial waves. This phenomenon is caused by the fact that when a surface wave travels, its propagation conditions change: displacements of particles in the medium at a certain time instant begin to reach the opposite side of the wedge, thus leading to splitting of a traveling wave to independent surface and spatial wave components. The energy of the surface wave begins to decrease during its propagation, because spatial waves that continuously travel from the surface deep into the medium carry away a part of the energy. These effects eventually lead to splitting of a Rayleigh wave into independent symmetric and antisymmetric modes.

In most cases, nonlinear effects are associated with changes in the properties of the medium in which oscillations propagate. Changes in the wave characteristics in the WSP are caused not by the influence of the medium but by the interaction of a propagating wave with wedge faces. An acoustic-field traditionally arises at high intensities of waves; however, an nonuniform field structure in a WSP manifests itself even at low intensities. The acoustic field in a WSP is actually a superposition of two nonuniform waves that travel in the forward and backward directions. Therefore, our study consists of two stages. First, we consider the evolution of a surface wave that travels perpendicular to the wedge edge, and then the appearing interference of two counterpropagating wave flow is analyzed.

EVOLUTION OF A RAYLEIGH WAVE IN A WEDGE-SHAPED PLATE

The energy of a surface wave that travels in a WSP does not remain constant but decreases monotonically owing to continuous energy transfer to body waves. Therefore, the classical system of equations of wave dynamics, which is usually deduced from the law of energy conservation without consideration for dissipation processes, can hardly be used to solve the problem. Let us consider an approach that allows one to approximately estimate the evolution of a traveling surface wave using the plane-wave model. In a cylindrical coordinate system that allows partitioning of the Lame vector equation for each wave potential, the standard dynamics equations have the form [7]

\[
\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + k_i^2 \Phi = 0, \\
\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} + k_i^2 \Psi = 0.
\] (1)

The presence of the first derivative with respect to the longitudinal coordinate in (1) indicates that the
amplitude of the potentials increases toward the edge because of wave-flow compression in the WSP. However, these equations disregard both the change in amplitude caused by the energy loss during transformation of the surface wave into spatial waves and the change in the wave velocity.

The problem cannot be solved by introducing of additional term into Eqs. (1) that would take the amplitude and wave-velocity changes into account, because it is very difficult to determine the changing parameters as functions of coordinates. If it is assumed that these parameters slowly change within a local wavelength, Eqs. (1) can be used to obtain an approximate solution. In small intervals of changes in \( r \) and \( t \), solutions of a general form can be considered as those consisting of elementary solutions that have the form

\[
\Phi(r) = A_0 \exp[ik(r) - i\omega t] \sin(\nu \theta),
\]

\[
\Psi(r) = A_0 \exp[ik(r) - i\omega t] \sin(\eta \theta),
\]

where \( k(r) = k_0(1 + ar), \) \( dk(r)/dr = k_0a = 2\pi a/\lambda, \) \( a = \varepsilon \lambda, \) and \( \varepsilon \ll 1, \) where \( \varepsilon \) is a small parameter of the problem. The wavenumber in these relationships is expressed in complex form: \( k(r) = k_0(r) + k_x(r), \) where the real part of this number determines the wave velocity \( V(r) = \omega/k_0(r) \) and the imaginary part determines the amplitude \( A(r) = A_0 \exp(-k_x(r)). \) The same approach applies to the boundary conditions that express the absence of stresses at the wedge surfaces:

\[
\sigma_{r\theta} = \sigma_{\theta\theta} = 0, \quad at \quad \theta = \pm \theta_0,
\]

where

\[
\sigma_{r\theta} = \rho c^2 \left[ \frac{2\partial^2 \Phi}{r \partial r \partial \theta} - \frac{2\partial \Phi}{r^2} + \frac{1}{\partial \theta^2} \right],
\]

\[
\sigma_{\theta\theta} = 2\rho c^2 \left[ \frac{k_0^2}{2} \partial^2 \Phi - \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{\partial \theta^2} \right].
\]

It is desirable to seek solutions of (1) in the form of combinations of Hankel functions of the first kind:

\[
\Phi = \left[ A_0 \mathcal{H}^{(1)}_0(kr) \cos(\nu \theta) + C_0 \mathcal{H}^{(3)}_0(kr) \sin(\nu \theta) \right] \exp(-i\omega t),
\]

\[
\Psi = \left[ A_0 \mathcal{H}^{(1)}_0(kr) \sin(\eta \theta) + C_1 \mathcal{H}^{(3)}_0(kr) \cos(\eta \theta) \right] \exp(-i\omega t),
\]

because these functions satisfy the cancellation conditions. In these expressions, the angular wavenumbers \( \nu \) and \( \eta \) are related to the sought wave numbers and are assumed to slowly change on a local wavelength. Substituting relationships (4) into boundary conditions (3) leads to a system of four algebraic equations with respect to the constants \( A_0, A_1, C_0, \) and \( C_1. \) Using the relationships for cylindrical functions \( [8] \)

\[
2\nu Z_\nu(z) = z Z_{\nu-1}(z) + z Z_{\nu+1}(z),
\]

\[
2\nu Z_\nu(z) = Z_{\nu-1}(z) - Z_{\nu+1}(z),
\]

after simple transformations, we obtain

\[
H_\nu^{(0)}(r) - \frac{1}{\rho} H_\nu^{(1)}(r) = \frac{\rho}{4\nu \lambda} \left[ H_{\nu-2}^{(0)}(r) - H_{\nu-2}^{(1)}(r) \right] = \frac{\rho}{4\nu} H_{\nu-2}^{(1)},
\]

\[
H_\nu^{(0)}(r) + \frac{1}{\rho} H_\nu^{(1)}(r) = \frac{1}{4} \left[ H_{\nu-2}^{(0)}(r) + H_{\nu-2}^{(1)}(r) \right] = \frac{1}{4} H_{\nu-2}^{(1)},
\]

where \( H_\nu^+ = H_{\nu-2}^{(0)}(r) + H_{\nu-2}^{(1)}(r), \) \( H_\nu^- = H_{\nu-2}^{(0)}(r) - H_{\nu-2}^{(1)}(r). \) Primes denote the first and second derivatives with respect to the argument, respectively. Taking these transformations into account, we obtain the complex dispersion equation for determining the wave amplitude and velocity:

\[
\frac{H_\nu^+}{H_\nu^0(kr)} - \frac{2(k_n^2 - 1)}{H_\nu^0(kr)} \left[ \frac{\tan \eta \theta}{\tan \nu \theta} \right] = 0,
\]

where \( H_\nu^+ = H_{\nu-2}^{(0)}(x_j) \pm H_{\nu-2}^{(1)}(x_j), \) \( H_\nu^- = H_{\nu+2}^{(0)}(x_j) \pm H_{\nu+2}^{(1)}(x_j), \) \( x_1 = k_0 r, x_2 = k_0 r, n = \pm 1. \)

**Numerical Simulation**

The spectral characteristics of surface waves are obtained numerically from the solution to Eq. (7). The initial data for this problem are the wave parameters (oscillation frequency and initial amplitude) and the properties of the medium: the elastic constants, density, and dimensions of the WSP. At a given time instant, the value of the wave vector must be calculated at a specified distance from the observation point. The Hankel functions are calculated using the Langer asymptotic representation \([8],\) which is uniformly satisfied within the interval \( 0 < x < \infty \) for large values on the order of

\[
H_\nu^{(0)}(x) \approx \sqrt{\lambda} \exp \left( -\frac{2\pi i}{3} \right) \mathcal{H}^{(2)}_{\nu 3} \left( e^{2\pi i \nu \lambda} \right),
\]

\[
\omega = \sqrt{1 - \frac{x^2}{v^2}}, \quad \lambda = \frac{1}{\omega} \text{Arth} \omega - 1,
\]

and the Hankel asymptotic expansion for large values of the argument

\[
H_\nu^{(0)}(z) \approx \sqrt{\frac{2}{\pi z}} \exp \left[ -i \left( z - \frac{\nu \pi}{6} - \frac{\pi}{4} \right) \right],
\]

which, in view of the adopted designations, take the form

\[
H_\nu^{(0)}(k r) \approx \sqrt{\frac{2}{\pi k}} r^{-0.5} \exp \left[ -ikr - i\pi \arctan \left( \frac{\pi}{k r} \right) \right],
\]

\[
H_\nu^{(1)}(k r) \approx \sqrt{\frac{2}{\pi k}} r^{-0.5} \exp \left[ -ikr - i\pi \arctan \left( \frac{\nu}{k r} \right) \right].
\]

**ACOUSTICAL PHYSICS** **Vol. 59** **No. 4** 2013