CRystallographic Symmetry

Using the $P$-Symmetry Space Groups of Crystals to Investigate 6D Symmetry Groups

A. F. Palistrant
State University of Moldova, ul. Mateevicha 60, Chisinau, Moldova
e-mail: mepalistrant@yandex.ru

Received March 26, 2008

Abstract—All $P$ symmetry space groups of crystals in the geometric classification that are allowed by the general theory of $P$ symmetry have been investigated. The number of different symmetry groups of a 6D Euclidean space that retain the invariant 3D plane has been revealed using the different (disregarding enantiomorphism) symmetry space groups and crystallographic $P$ symmetries generated by these groups.

PACS numbers: 61.50.Ah
DOI: 10.1134/S1063774509040014

Introduction and Formulation of the Problem

Deriving and analyzing multidimensional symmetry groups $n$-dimensional Fedorov groups $G_n$, their flat subgroups $G_{nm} (n > m)$ with an invariant $m$-dimensional plane; and $G_{nm,k} (n > m > ... > k)$ with invariant $m$-dimensional, ..., and $k$-dimensional planes successively embedded into each other (a straight line and a point are considered 1D and zero-dimensional planes, respectively) are urgent problems of multidimensional geometric crystallography.


One especially fruitful approach for multidimensional $P$ symmetry applications is geometrically classifying $P$ symmetries [9, 7], which makes it possible to describe (using 3D point groups $G_3^p$ of these 32 $P$ symmetries) the category $G_{630}$ of 6D point symmetry groups with an invariant 3D plane [10, 11] and (using the rosette ($G_{230}^p$) tablet ($G_{320}^p$), frieze ($G_{21}^p$), and ribbon ($G_{321}^p$) groups of these $P$ symmetries) the category of 5D and 6D plane-point and plane-linear symmetry groups $G_{520}$, $G_{6320}$, $G_{521}$, and $G_{6321}$ [7]. Furthermore, the geometric principle of classification was extended to the rosette, tablet, and hypertablet $P$ symmetries [12].

This study is devoted to the use of space groups $G_3^p$ of crystallographic $P$ symmetries to reveal the number of different $G_{63}$ groups of a 6D Euclidean space retaining an invariant 3D plane in it.

Some Statements of the Theory of $P$ Symmetry

$P$ symmetry, developed by Zamorzaev [13, 4, 7], is a profound generalization of the classical theory of symmetry [14, 15]. In essence, $p$ symmetry differs from Shubnikov’s antisymmetry [1, 2] by the arbitrary number of $p$ properties assigned to figure points; it differs from Belov’s $p$-color symmetry [3, 4] by the arbitrariness of the group $P$ of the substitutions of properties assigned to the points of a figure at its isometric transformations. $P$ symmetry covers all the generalizations of Shubnikov’s antisymmetry and Belov’s color symmetry, where the law of change in the properties assigned to the points of the figure under consideration is combined directly with the isometric transformation of this figure, which do not affect the properties assigned to its points and act only on the points of the figure considered and are not related to the choice of parts of this figure (in contrast to the Wittke—Garrido polychromatic symmetry [16], which is not covered by $P$ symmetry).

The set of all $P$-symmetry transformations of a figure $F$, with properties assigned to its points and denoted by subscripts $i = 1, 2, ..., p$ or “+” or “−” signs, forms a multiplicative group $G$ with respect to the multiplication of $P$-symmetry transformations. Thus, any transformation $g$ from the group $G$ is expanded in a commutative product of $s$-symmetry
transformation of the transformed figure F and the
substitution $\varepsilon = \begin{pmatrix} 1 & 2 & \ldots & p \\ k_1 & k_2 & \ldots & k_p \end{pmatrix}$ of $p$ properties

assigned to the points of this figure ($g = se = \varepsilon s$). The set of all $s$-symmetry transformations entering $g$
morphisms of group $G$ form its generating group $S$, and
the substitutions of indices $s$ form group $P$. At $P = \bar{P}$
or $e \subset P \subset P$, we will refer to group $G$ as a complete or
incomplete $P$-symmetry group, respectively.

If $G$ is a complete $P$-symmetry group, $H = G \cap S$ is
its symmetry subgroup and $Q = G \cap P$ is a subgroup
of index substitutions. The group $G$ will be referred to as
senior at $Q = P$ (then $H = S$ and $G = S \times P$), junior
at $Q = e$ (then $G$ is isomorphic to $S$, which
isomorphic to $S$, which
leads us to the symbolic recording $G \cong S$), and the $Q$-medium
group at $e \subset Q \subset P$. Any group $G$ of complete $P$
symmetry, as indicated in Zamorzaev’s main theorem
[4, 7], can be derived from its generating group $S$ by
finding normal dividers $H$ and $Q$ in $S$ and $P$ (allowing
isomorphism of the factor group $S/H$ on $P/Q$) by mul-
tiplying the isomorphism-relevant adjacent classes and combining the obtained products.

Deriving the senior $P$-symmetry groups is unim-
portant, because they correspond to the $Q = P$. There-
fore, the factor group $S/H$ becomes isomorphic on
$P/Q$ only when the normal divider $H$ of the group
$S$ coincides with it; this means that the senior group $G$
of $P$ symmetry is a direct product of the generating
$S$ and substitution group $P$, which characterizes
the $P$ symmetry under consideration ($G = S \times P$). The junior groups of the given $P$ symmetry are derived from
a certain generating group $S$ (according to the funda-
mental theorem) only when $S$ has a normal divider $H$
so that $S/H \cong P$, because $Q = e$ for this type of $P$
symmetry group. In practice, junior $P$ symmetry groups
(for which $P(Q = P)$ are usually derived by the Shub-
nikov—Zamorzaev method: alternately replacing the
symmetry transformations of the initial group $S$ in
the system of its generators with the corresponding
$P$-symmetry transformations so as to provide isomor-
phism of the new group $G$ obtained, in this case, from
the taken group $S$ and make the index substitutions
entering the $G$-group transformations constitute the
noted group $P$. This method was used in [17] to derive
point groups of magnetic crystal symmetry and in [18]
to derive space groups of magnetic crystal symmetry.

The study of $Q$-medium groups of $P$ symmetry,
according to the above-mentioned fundamental the-
orem, is related to the enumeration of nontrivial normal
dividers $Q$ of substitution groups $P$, which characterize
the $P$ symmetries under consideration; these groups
can be calculated if junior $P$-symmetry groups are pre-
liminarily revealed, because, as was shown in [19],
the number of $Q$-medium $P$ symmetry groups in a given
set is equal to the number of junior $P_{0}$-symmetry
groups with the same generator if the factor group $P/Q$
is significantly isomorphic to $P_{0}$ ($P/Q \cong P_{0}$).

It follows from the description of $P$ symmetry that
there is an infinite set of $P$ symmetries, because no
limitations are imposed on the group $P$ of index substi-
tutions specifying the $P$ symmetry. In the $P$-symmetry
scheme, Shubnikov’s antisymmetry is a 2-symmetry
characterized by the group $P = \{1, 2\}$ (or $\perp$-symmetry,
specified by the group $P = \{1, -\}$) [2].
Zamorzaev’s antisymmetry of different types ($l$-multi-
ple) is considered (2, ..., 2)-symmetries (where the number 2 is repeated $l$ times) and is specified by
the group $P$, coinciding with the Abelian group $E_{l} = \{e_{1}\} \times \{e_{2}\} \times \ldots \times \{e_{l}\}$, which is expanded in the direct
product of $l$ second-order groups generated by the
antiidentical transformations $e_{i}$ of $l$th types ($i = 1, 2, \ldots, l$)
[2]. Belov’s color symmetry, referred to as the $p$ sym-
metry in [4, 7], corresponds to the cyclic group $P = \{1, 2, \ldots, p\}$, while Poly’s color antisymmetry, known as the $(p/)$
symmetry, is specified by the group $P = \{(1, ..., p)(\bar{p}, ..., \bar{1}), (1, \bar{1})...(p, \bar{p})\}$ with $2p$ trans-
formed indices: $p$ positive ones $(i)$ and $\bar{p}$ negative ones
($\bar{i}$) [4, 7].

Synthesizing the concepts of $p$ and $(p/)$ symmetries
with the (2, ..., 2) symmetry led Neronova and Belov [8]
to the representation of color antisymmetry (or $(p, 2)$
symmetry [4]). Based on this synthesis, Zamorzaev and
colleagues (Faculty of Mathematics and Informatics,
Moldova University) introduced the concepts of color
antisymmetry of different types (or $p$, 2, ..., 2) sym-
metry) and $(p/)$ antisymmetry, both simple (or $(p/)$, 2) sym-
metry) and multiple (or $(p/2, ..., 2)$ symmetry) [4, 7].

In the $P$ symmetry scheme, $(p, 2)$ symmetry is set
by the substitution group $P = \{(1, 2, ..., p) \} \times \{(+, -)\} = \{(1, , 2+, ..., p+)\} \times \{(1, -, 2-, ..., p-)\}$
and the $(p/2, 2)$ symmetry is set by the group $P = \{(1, 2, ...
, p)(\bar{p}, ..., \bar{2}, \bar{1}, (1, \bar{1})...(p, \bar{p})\} \times \{(+, -)\} = \{(1+,
2+, ..., p+)(\bar{p}+, ..., \bar{2}+, \bar{1}+\}(1, -, 2, ..., p-)(\bar{p}-, ..., \bar{2}-, \bar{1}-\}(1, +, \bar{1}+)...(p+, \bar{p}+\}(1-,..., \bar{p}-, \bar{2}-, \bar{1}-\)...
(1+,...,$(p+,...)$($\bar{1}+,$ $\bar{1}-...($p+$,$ \bar{p})$ etc).

The above-mentioned particular cases of $P$ sym-
metry have a simple descriptive geometric interpreta-

For example, index substitution groups $P$ in the
cases of 2, (2, 2), $p$, and $(p/)$ symmetries are presented,
respectively, by substitutions of the numbers of vertices of
a segment, rectangle, oriented regular $p$-gon, and equi-
angle-semiregular $2p$-gon at their symmetry transfor-
mations, while the $(p, 2)$ and $(p/2, 2)$ symmetries are
represented by the substitutions of the numbers of vertices
of a regular prism with equally oriented $p$-angle bases and
an equiangle-semiregular prism with $2p$-angle
bases [4, 7].

Note that the substitution group $P$ that character-
izes each of the noted $P$ symmetries is isomorphic to
the group $C_{2}$ at 2-symmetry, the group $C_{2v}$ at (2, 2)
symmetry, the group $C_{p}$ at $p$ symmetry, the dihedral
group $D_{p} (\cong C_{p})$ at $(p/)$ symmetry, the group $C_{ph}$ at