1. INTRODUCTION

Recent investigations of the strong coupling regime in $\phi^4$ theory revealed an unexpected feature in its renormalizability: the continual limit in the renormalized theory does not require the continual limit in the bare theory. We show below that such a kind of renormalizability has a general character and can be understood in the framework of Wilson's many-parameter renormalization group. These results make it possible to give a final solution to the problem of triviality or nontriviality of $\phi^4$ theory. Application of these ideas to QCD shows that Wilson’s theory of confinement is not purely illustrative, but has a direct relation to a real situation. As a result, the problem of analytical proof of confinement and a mass gap can be considered solved, at least on the physical level of rigor.

2. CHARACTER OF RENORMALIZABILITY IN $\phi^4$ THEORY

According to recent results [1–4] (see also [5, 6]), the Gell-Mann–Low function $\beta(g)$ in four-dimensional $\phi^4$ theory is nonalternating and has a linear asymptotics at infinity. According to the Bogoliubov and Shirkov classification, it means the possibility of constructing a continuous theory with finite interaction at large distances. This conclusion is in visible contradiction to the lattice results indicating triviality of $\phi^4$ theory (see [8–12] and numerous references in [13]).

In fact, we should differentiate two definitions of triviality. According to Wilson [8], triviality means that integration of the Gell-Mann–Low equation in the direction of large distances $L$ gives an effective charge $g$ tending to zero (Fig. 1a); this definition implies massless theory, since in the opposite case the distance scale is saturated by the inverse mass. The definition of true triviality is different (Fig. 1b). In this case we consider the massive theory and suggest the finite interaction $g_{\infty}$ for $L \approx m^{-1}$; a theory is trivial if integration of the Gell-Mann–Low equation in the direction of small $L$ gives a divergency at finite $L_0$ (the so-called Landau pole) and does not allow the $L \to 0$ limit to be reached. Such a situation is internally inconsistent [7] and signifies incorrectness of the initial suggestion on finite interaction at large distances; in fact, $L_0 \to 0$ if $g_{\infty} \to 0$. Wilson triviality means that the $\beta$-function is nonnegative and has a zero only for $g = 0$. True triviality needs in addition its sufficiently quick growth at infinity, $\beta(g) \propto g^\alpha$ with $\alpha > 1$. According to [1–6], $\phi^4$ theory and QED are trivial in the Wilson sense, but do not possess true triviality.

Two definitions of triviality were hopelessly mixed in literature [13]. The reasons for it are as follows:

(a) Bogoliubov and Shirkov’s work is poorly known to the Western community.
(b) It is rather difficult to test true triviality in the lattice approach.

(c) There exist arguments that "prove" the equivalence of two definitions.

As an illustration to the latter point, consider the following reasoning. The only alternative to the perturbative approach is to express all quantities related to renormalized theory in terms of functional integrals. The latter depend on the bare charge $g_0$, bare mass $m_0$, and the ultraviolet cut-off $\Lambda$. Taking into account their dimensional character, we have the following relations for the renormalized charge $g$, renormalized mass $m$, and observable quantities $A_i$:

$$g = F_\Lambda(g_0, m_0/\Lambda), \quad m = \Lambda F_m(g_0, m_0/\Lambda),$$

$$A_i = \Lambda^{d_i} F_i(g_0, m_0/\Lambda),$$

where $d_i$ is a physical dimensionality of $A_i$. Excluding $g_0$ and $m_0/\Lambda$ in favor of $g$ and $m/\Lambda$, we have

$$A_i = m^{d_i} F_i(g, m/\Lambda).$$

To eliminate the dependence on $\Lambda$ we should take the limit $m/\Lambda \rightarrow 0$. In the lattice approach, this limit corresponds to $\xi/a \rightarrow \infty$ ($\xi$ is the correlation length and $a$ is lattice spacing), i.e., to the phase transition point. The latter is determined by a zero of the $\beta$-function, which gives $g = 0$ in four-dimensional $\phi^4$ theory.

In this argumentation, Wilson triviality was considered as given, while true triviality was "derived" from it. Of course, it cannot be correct, because two definitions are surely not equivalent. This shortcoming originates from our assumption that a general-position situation takes place in Eq. (4); in this case we indeed should take the limit $m \rightarrow 0$. This limit is unnecessary if dependence on $m/\Lambda$ is absent in Eq. (4). Such a special case fills the "gap" between the two definitions and renders the inequivalent.

Such a special case actually holds in $\phi^4$ theory [3, 4]. Let us return to Eqs. (3) and impose the condition $m \ll \Lambda$, corresponding to the continuum limit of the renormalized theory. If this condition is imposed in the region $g_0 \gg 1$, then $\phi^4$ theory reduces to the Ising model, containing the single parameter $\kappa$, which plays the role of inverse temperature [3, 4]; relations (3) take the form

$$g = F_\kappa(\kappa), \quad m = \Lambda F_m(\kappa),$$

$$A_i = \Lambda^{d_i} F_i(\kappa).$$

So far there is nothing unusual: the condition $m/\Lambda \rightarrow 0$ gives a relation between $g_0$ and $m_0/\Lambda$, so all functions in Eq. (3) depend on the single parameter, which we denoted as $\kappa$. The nontrivial point consists in the following: the condition $m \ll \Lambda$ is sufficient for transformation to the Ising model, but not necessary for it. In fact, such transformation is possible under the weaker conditions, which are compatible with the arbitrary value of $m/\Lambda$ [3, 4]. Excluding $\kappa$ from Eqs. (5), we obtain the equations

$$A_i = m^{d_i} F_i(g),$$

which are analogous to (4), but do not contain the parameter $m/\Lambda$. As a result, the program of renormalization is completely fulfilled and no additional limiting transitions are necessary. It means that (a) we can