Algebraic Form of the M3-Brane Action

H. Ghadjari\(^a\) and Z. Rezaei\(^b\)

\(^a\) Department of Physics, Amirkahr University of Technology (Tehran Polytechnic), Tehran, 15875-4413 Iran
\(^b\) Department of Physics, University of Tafresh, Tehran, 39518-79611 Iran

e-mail: h-ghajari@aut.ac.ir; z.rezaei@aut.ac.ir

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Abstract—We reformulate the bosonic action of an unstable M3-brane to manifest its algebraic representation. It is seen that in contrast to string and M2-brane actions, which are respectively represented only in terms of two- and three-dimensional Lie algebras, the algebraic form of the M3-brane action is a combination of four-, three-, and two-dimensional Lie algebras. Corresponding brackets appear as mixtures of the tachyon field, space–time coordinates \(X\), the two-form field \(\omega^{(2)}\), and the Born–Infeld one-form \(b_\mu\).

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1. INTRODUCTION

Algebraic reformulation of known actions in string theory and M-theory shows that string theory is based on the conventional algebra, or a two-dimensional Lie algebra (known as a two-algebra), but a complete description of M-theory requires an extended Lie algebra called a three-algebra \([1]\), which was mainly developed in \([2–5]\). The numbers two and three are respectively associated with string theory and M-theory. Two is the string worldsheet dimension and also the codimension of D-branes in both type-IIA and type-IIB superstring theories \([6]\). Three is the membrane worldvolume dimension in M-theory and the codimension of M2- and M5-branes. This means that via two-algebra interactions, some \(D^p\)-branes can condense to a \(D^{(p+2)}\)-brane \([7]\) and through three-algebra interactions multiple M2-branes condense to a M5-brane \([8–16]\). These connections between two and three and, respectively, string theory and M-theory become obvious by rewriting Nambu–Goto actions in algebraic form.

By analogy, we can expect to describe \(p\)-branes applying a \((p+1)\)-algebra structure \([17]\). These extended algebras are used to construct worldvolume theories for multiple \(p\)-branes in terms of Nambu brackets that are classical approximations to multiple commutators of these algebras \([18]\). Nambu \(n\)-brackets introduce a way to understand the \(n\)-dimensional Lie algebra presented in \([19]\). Formulation of the \(p\)-brane action in terms of a \((p+1)\)-algebra makes it more compact and we are left with algebraic calculations, which are then usually simpler to handle.

In string theory, we are inevitably faced with unstable systems, and studying them deepens our understanding of the string theory. In bosonic string theory, the instability is always present due to the tachyon presence in the open string spectrum. Two examples of unstable states in superstring theories are non–BPS branes (odd (even) dimensional branes in type-IIA (IIB) theory) and brane–antibrane pairs in both type-IIA and type-IIB theories \([20, 21]\). An interesting fact about the dynamics of these unstable branes, generally obvious in the effective action formulation, is their dimensional reduction through tachyon condensation \([22–27]\). During this process, the negative energy density of the tachyon potential at its minimum point cancels the tension of the D-brane (or D-branes) \([28]\), and the final product is a closed-string vacuum without a D-brane or stable lower-dimensional D-branes.

On the other hand, stable objects in string theory can be obtained by dimensional reduction of stable branes in M-theory (M2- and M5-branes). Naturally, we can expect to have a preimage of unstable branes in superstring theories by formulating an effective action for unstable branes in M-theory. Among different unstable systems in M-theory \([29]\), the M3-brane is noteworthy because it is directly related to the M2-brane. Tachyon condensation of the M3-brane effective action results in the M2-brane action, and its dimensional reduction also leads to a non–BPS D3-brane action in type-IIA string theory \([30]\).

Despite attempts made to formulate the M3-brane action consistent with desired conditions \([30]\), there has been no algebraic approach towards this formulation. The existence of the algebraic form for the action of the M2-brane, as the fundamental object of M-theory, motivated us to search for the algebraic presentation of the M3-brane as the main unstable object in M-theory, whose instability is due to the presence of the tachyon.

What distinguishes the present study from conventional algebraic formulations is the instability of the
M3-brane. In other words, the presence of the tachyon and other background fields affect the resultant algebra. It is shown that a pure four-algebra does not occur, as expected, and we are encountered with four-, three-, and two-brackets that are mixtures of the tachyon, space–time coordinates, and other fields.

2. ALGEBRAIC M3-BRANE ACTION

The conventional action corresponding to a non-BPS M3-brane is a combination of the DBI (Dirac–Born–Infeld) and WZ (Wess–Zumino) parts [30]

\[ S = S_{\text{DBI}} + S_{\text{WZ}}, \]

\[ S_{\text{DBI}} = -\int dt^4 \xi V(T) \sqrt{|\hat{k}|} \sqrt{\det H_{\mu \nu}}, \]

\[ S_{\text{WZ}} = -\int dt^4 \xi V(T) e^{\hat{A}_\mu} \hat{T} \hat{S}_{\text{WZ}} \mathcal{K}_{3jkl}, \]

where \( \xi^\mu \) with \( \mu = 0, 1, 2, 3 \) labels worldvolume coordinates of the M3-brane, \( V(T) \) is the tachyon potential, which is an even function of \( T \) and is characterized as \( V(T = \pm \infty) = 0 \) and \( V(T = 0) = \mathcal{T}_{\text{M3}} \), \( \mathcal{T}_{\text{M3}} \) is the M3-brane tension, and \( \hat{K}^M(X) \) is the Killing vector, such that the Lie derivatives of all target-space fields vanish with respect to it [30]. Other fields in (2.1) are defined as

\[ H_{\mu \nu} = \hat{g}_{MN} \hat{D}_\mu \hat{X}^M \hat{D}_\nu \hat{X}^N + \frac{1}{|\hat{k}|} \hat{F}_{\mu \nu} + \frac{1}{|\hat{k}|} \partial_\mu T \partial_\nu T, \]

\[ \hat{k}^2 = \hat{k}^M \hat{k}_M = |\hat{k}|^2, \]

\[ \hat{F}_{\mu \nu} = \partial_\mu \hat{b}_\nu - \partial_\nu \hat{b}_\mu + \partial_\mu \hat{X}^i \partial_\nu (i_\nu \hat{C})_{MN}, \]

\[ \hat{D}_\mu \hat{X}^M = \partial_\mu \hat{X}^M - \hat{A}_\mu \hat{K}^M, \]

\[ \hat{A}_\mu = \frac{1}{|\hat{k}|^2} \partial_\mu \hat{X}^M \hat{K}^M, \]

\[ \hat{K}_{3jkl} = \partial_{\mu_1} \hat{C}^{(2)}_{\mu_2 \mu_3 \mu_4} - \partial_{\mu_1} \hat{C}^{(2)}_{\mu_2 \mu_3 \mu_4} + \partial_{\mu_2} \hat{C}^{(2)}_{\mu_1 \mu_3 \mu_4} + \frac{1}{3!} \hat{C}_{KMN} \hat{D}_\mu \hat{X}^K \hat{D}_\nu \hat{X}^M \hat{D}_\rho \hat{X}^N + \frac{1}{21} \hat{A}_{\mu_1} (\partial_\mu \hat{b}_{\mu_2} - \partial_\mu \hat{b}_{\mu_3}). \]

The tensor \( H_{\mu \nu} \) consists of the pullback of the background metric, the field strength \( \hat{F}_{\mu \nu} \) of the gauge field \( \hat{A}_\mu \), and the tachyon field \( \hat{T} \); \( M \) and \( N \) represent space–time indices and \( \hat{D}_\mu \) is the covariant derivative. The field strength itself is expressed in terms of the Born–Infeld 1-form \( \hat{b}_\mu \) and the R–R sector field \( \hat{C} \). The curvature of the 2-form \( \hat{\omega}^{(2)} \) is denoted by \( \hat{\kappa} \).

The determinant of the tensor \( H_{\mu \nu} \) in the DBI action can be decomposed as

\[ \sqrt{-\det H_{\mu \nu}} = \sqrt{-\det (G_{\mu \nu} + \hat{F}_{\mu \nu})}, \]

where

\[ \hat{F}_{\mu \nu} = \partial_\mu \hat{b}_\nu - \partial_\nu \hat{b}_\mu, \]

\[ \hat{G}_{\mu \nu} = L_{MN} \partial_\mu \hat{X}^M \partial_\nu \hat{X}^N + \frac{1}{|\hat{k}|} \partial_\mu \hat{T} \partial_\nu T, \]

and

\[ L_{MN} = g_{MN} + i_{\hat{F}} \hat{C}_{CMN} \frac{\hat{K} M N}{|\hat{k}|}. \]

Regarding (2.3), the DBI action can be expanded to the quadratic order [31] as

\[ S_{\text{DBI}} = -\int dt^4 \xi V(T) \sqrt{|\hat{k}|} \sqrt{-\det \hat{G}_{\mu \nu}} \]

\[ \times \left( 1 + \hat{F}_{\mu \nu} \hat{F}^{\mu \nu} + \ldots \right). \]

2.1. DBI Part of the M3-Brane Action

To find the algebraic form of the DBI action, we start with the first term in (2.6), \( \sqrt{-\det \hat{G}_{\mu \nu}} \), which is the determinant of a \( 4 \times 4 \) matrix and all its elements are sums of a tachyonic part and a space-like part \( (\partial \hat{X} \partial \hat{X} + \partial \hat{T} \partial \hat{T}) \). This determinant totally consists of \( 48 \times 8 \) terms. These terms can be classified into sixteen \( 4 \times 4 \) determinants such that the elements of these determinants are only \( \partial \hat{X} \partial \hat{X} \) or \( \partial \hat{T} \partial \hat{T} \) and not sums of them. Hence, each determinant has 24 terms such that adding them leads to the same number of terms (16 \( \times 24 \)) as in the initial main determinant. These 16 determinants can be categorized as: one determinant with \( \partial \hat{X} \partial \hat{X} \) elements (four combinations from the 4 states \( \binom{4}{4} = 1 \)), one determinant with elements \( \partial \hat{T} \partial \hat{T} \binom{4}{4} = 1 \), four determinants with three rows of \( \partial \hat{X} \partial \hat{X} \) elements and one row of \( \partial \hat{T} \partial \hat{T} \) elements \( \binom{4}{1} = 4 \), four determinants with three rows of \( \partial \hat{T} \partial \hat{T} \) elements and one row of \( \partial \hat{X} \partial \hat{X} \) elements \( \binom{4}{1} = 4 \), and, finally, six determinants with two rows of \( \partial \hat{T} \partial \hat{T} \) elements and two rows of \( \partial \hat{X} \partial \hat{X} \) elements \( \binom{4}{2} = 6 \). It follows that determinants with more than one row of \( \partial \hat{T} \partial \hat{T} \) are zero. We are therefore left with two kinds of determinants: a determinant consisting of only \( \partial \hat{X} \partial \hat{X} \) entities and those with three rows of \( \partial \hat{X} \partial \hat{X} \) elements and one row of \( \partial \hat{T} \partial \hat{T} \) entities. Because