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On The Motion of Charged Particles in an Alternating Nonuniform Electric Field

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Abstract—The equations of motion for a charged particle in an electric field featuring a stationary and an oscillating component are considered for the case where the force of friction is linear in the particle velocity. The averaging of these equations over the period of field oscillations is legitimate under some specific conditions. The most general expression for an additional stationary force acting on the particle under these conditions is derived, and the limiting values of this force are found. Applications of the results obtained in the present study are considered. Such applications include the use of pulsed currents in the electrochemical dimensional treatment of materials.

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INTRODUCTION

It is well known [1] that a charged particle in an electric field of the form $E(t)\sin(\omega t + \vartheta)$ experiences the action of the stationary force

$$F(r) = -\frac{q^2}{4m\omega^2} \nabla E^2(r). \quad (1)$$

This force, referred to as the Gaponov–Miller force [2] or the Miller force [3], induces a number of nonlinear effects in plasmas [24]. In plasma physics, it is called a high-frequency-pressure force.

A generalization of the force in (1) to the case of particle motion in a viscous medium where the force of viscous friction has the form $F = -mv$ was given in [5]. The result is

$$F = -\frac{q^2}{2m(\omega^2 + \nu^2)} (\nabla \cdot \mathcal{E}) \cdot \mathcal{E}. \quad (2)$$

In the case of $\nabla \times \mathcal{E} = 0$, the high-frequency limit of this expression coincides with expression (1), while its low-frequency limit is frequency-independent. It is this limit that was used in [5] to explain the mechanism of a sharp increase in the breakdown voltage in high-voltage facilities of mains frequency upon going over from stationary to alternating fields.

The effect of an oscillating perturbation on mechanical systems was also examined in [6]. If a particle moves simultaneously in stationary and oscillating fields, the equation of motion for this particle has the form

$$m\ddot{x} = F(x) + f(x)\sin \omega t, \quad (3)$$

where $x(t)$ is a generalized coordinate. Representing it as the sum $x(t) = X(t) + \xi(t)$, where $X(t)$ describes a “smooth” motion of the particle and $\xi(t)$ describes its oscillations about the smooth motion, and averaging Eq. (3) over the period of oscillations, we obtain

$$m\ddot{X} = F(X) - \frac{1}{4m\omega^2} \frac{d}{dX} \ddot{X}(X). \quad (4)$$

Thus, we have seen that, in the system being considered, there appears yet another stationary force whose form coincides with the form in (1) and which changes the physical properties of the system. An example that is presented in [6] and which is associated with an inverted pendulum is of interest: if the pivot of a simple pendulum (a pointlike mass on a massless rod) executes vertical oscillations, not only the lower but also the upper position of the mass (inverted pendulum) may be stable. The investigation of an inverted pendulum for stability in [7] yielded the following results: for the case where the pendulum length is $l = 20$ cm and where the amplitude of pivot oscillations is $a = 1$ cm, the upper position is stable if the frequency of pivot oscillations satisfies the condition $\omega > 43\omega_0$, where $\omega_0$ is the frequency of the pendulum being considered. This is the answer to the nontrivial question of the lower limit on the perturbation frequency at which the concept of a stationary additional force is valid.

Relying on the method proposed in [5], we will show that a further extension of the above results is possible.

1. OSCILLATING FIELD

The equation of motion for a particle in an electromagnetic field and in the presence of the force of friction linear in the particle velocity has the form

$$m\ddot{r} + m v\dot{t} = qE + \frac{q}{c} \mathcal{B}. \quad (5)$$
If the electric field changes as
\[ \mathcal{E}(r, t) = \mathcal{E}(r) \sin(\omega t + \varphi), \]
then, since the electric and magnetic fields are related by the equation
\[ \nabla \times \mathcal{E} + \frac{1}{c} \frac{\partial \mathcal{B}}{\partial t} = 0, \]
the magnetic field has the form
\[ \mathcal{B}(r, t) = \mathcal{B}(r) \cos(\omega t + \varphi). \]
This means that the field amplitudes are related by the equation
\[ \nabla \times \mathcal{E}(r) - \frac{\omega}{c} \mathcal{B}(r) = 0. \]

We assume that, within the time equal to the period of oscillations, the particle being considered travels a distance over which the field amplitudes change insignificantly. Expanding them in a series, we can then restrict ourselves to first-approximation terms; that is,
\[ \mathcal{E}(r) = \mathcal{E}(0) + (r \cdot \nabla)\mathcal{E}(0), \]
\[ \mathcal{B}(r) = \mathcal{B}(0) + (r \cdot \nabla)\mathcal{B}(0). \]

Substituting this expansion of the fields into Eq. (5) and taking into account Eq. (9), we obtain an equation of motion in the form
\[ m\ddot{r} + mv \dot{r} = q\mathcal{E}_0 \sin(\omega t + \varphi) + q(r \cdot \nabla)\mathcal{E}_0 \sin(\omega t + \varphi) \]
\[ + \frac{q}{\omega} r \times (\nabla \times \mathcal{E}_0) \cos(\omega t + \varphi). \]

In contrast to Eq. (5), this equation is linear, and we can solve it by the method of successive approximations. Assuming that the second and third terms on the right-hand side of it are small and discarding them for this reason, we obtain the equation of motion in the zero-order approximation; that is,
\[ m\ddot{r} + mv \dot{r} = q\mathcal{E}_0 \sin(\omega t + \varphi). \]

A particular solution of this equation is
\[ r(t) = -\frac{q\mathcal{E}_0}{m\omega^2 + v^2} \sin(\omega t + \varphi + \varphi_0). \]

Accordingly, we have
\[ \dot{r}(t) = -\frac{q\mathcal{E}_0}{m\omega^2 + v^2} \cos(\omega t + \varphi + \varphi_0), \]
where
\[ \tan \varphi_0 = v/\omega. \]

Substituting solutions (13) and (14) of Eq. (12) into the expression on the right-hand side of Eq. (11), averaging it over the period of oscillations, and taking into account the identity
\[ \nabla \mathcal{E}^2 = 2(\mathcal{E} \cdot \nabla)\mathcal{E} + 2\mathcal{E} \times (\nabla \times \mathcal{E}) \]
we obtain an equation of motion in the form (we must now omit the symbol “0” characterizing the beginning of the expansion in a series)
\[ m\ddot{r} + mv \dot{r} = \mathbf{F}(r), \]
where
\[ \mathbf{F}(r) = -\frac{q^2}{4m(\omega^2 + v^2)} \nabla \mathcal{E}^2(r). \]

The force in Eq. (18) generalizes the force in Eq. (2), coinciding with it at
\[ \nabla \times \mathcal{E} = 0. \]

A further generalization of the force in Eq. (2) is possible under the condition in (19).

## 2. OSCILLATING FIELD: INCLUSION OF INITIAL CONDITIONS

We now write the equation of motion for a particle in an alternating electric field (Eq. (5) without the Lorentz force) in the form
\[ m\ddot{r} + mv \dot{r} = q\mathcal{E}(r) \sin(\omega t + \omega) \]
and its first approximation in the form
\[ m\ddot{r} + mv \dot{r} = q\mathcal{E}_0 \sin(\omega t + \varphi) \]
\[ + q(r \cdot \nabla)\mathcal{E}_0 \sin(\omega t + \varphi). \]

The zero-order approximation of Eq. (21) has the form (12). Instead of the above particular solution, we now consider a general solution of Eq. (12) in the form
\[ r(t) = C_1 + C_2 \exp(-\nu t) \]
\[ -\frac{q\mathcal{E}_0}{m\omega^2 + v^2} \sin(\omega t + \varphi + \varphi_0). \]

Here, the constants of integration are determined from the initial conditions
\[ r(0) = 0, \quad \dot{r}(0) = v_0. \]

As a result, we arrive at
\[ C_1 = -C_2 + \frac{q\mathcal{E}_0}{m\omega^2 + v^2} \sin(\varphi + \varphi_0), \]
\[ C_2 = -\frac{v_0}{\nu} - \frac{q\mathcal{E}_0}{m\nu(\omega^2 + v^2)} \cos(\varphi + \varphi_0). \]

Substituting solution (22) of Eq. (12) into the expression on the right-hand side of Eq. (21) and averaging it over the period of oscillations, we obtain, for the right-hand side of Eq. (21), an intermediate expression (as before, we remove the subscript “0” on \( \mathcal{E} \) and \( \mathcal{B} \)) in the form
\[ \mathbf{F} = -\frac{q^2}{\omega^2 + v^2} (\mathcal{E} \cdot \nabla)\mathcal{E}(\omega \cos \varphi + v \sin \varphi) \]
\[ -\frac{q^2}{m(\omega^2 + v^2)} (\mathcal{E} \cdot \nabla)\mathcal{E}(\omega^2 \cos^2 \varphi - v^2 \sin^2 \varphi) \]
\[ -\frac{q^2}{2m(\omega^2 + v^2)} (\mathcal{E} \cdot \nabla)\mathcal{E}, \]
where