Self-Oscillating Systems with Chaotic Dynamics
Based on the Van der Pol–Duffing Equations

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Abstract—Mathematical models of the systems of the coupled van der Pol–Duffing equations with the chaotizing feedback algorithm (CFA) and the parametric chaotization algorithm (PCA) are considered. Numerical methods are used to analyze particular cases of the systems that do not employ the CFA and have mutual or unidirectional resistive coupling. The chaotic modes formed under asynchronous interactions of the oscillations of partial oscillators are considered. It is demonstrated that oscillations can be chaotized using the CFA or PCA when regular operating modes are realized in the absence of these algorithms.

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1. INTRODUCTION

The autonomous van der Pol–Duffing equation representing a simple modification of the classical van der Pol equation exhibits a simple dynamics. However, in the presence of an external harmonic signal, the system may exhibit complex oscillations outside the locking band or even chaos.

Simple and complex regular motions in the nonautonomous van der Pol–Duffing equation are studied in detail in [1]. There has been considerable recent interest in the analysis of transient processes [2]. Note the significantly smaller amount of works that have been devoted to chaotic modes of nonautonomous systems [3]. Mutual synchronization in the presence of various nonlinearities of the restoring force and the system of the coupled van der Pol–Duffing equations with the chaotizing feedback algorithm (CFA) and the parametric chaotization algorithm (PCA) remain unstudied.

In the context of extensive recent studies of various self-oscillating systems with chaotic dynamics, the analysis of systems based on the modified van der Pol equations is of significant interest since the van der Pol equation serves as a basis for a large number of models describing various phenomena in many fields of science.

In this study, we present mathematical models of the system of the coupled van der Pol–Duffing equations in which the CFA or PCA provides for chaos at various parameters. We consider particular cases of standard modes of the mutually and unidirectionally coupled van der Pol–Duffing equations that are used as starting equations for CFA and PCA systems.

2. MATHEMATICAL MODEL

A relatively general representation of the system of the linearly coupled van der Pol–Duffing equations is given by

\[ d^2 x_i/dt^2 - \varepsilon_i(1-x_i^2)dx_i/dt + \mu_ix_i + v_ix_i^n \]

\[ = \sum_{j, j \neq i} [\alpha_{ij} d^2 x_j/dt^2 + \beta_{ij} dx_j/dt + \delta_{ij}x_j + \gamma_{ij}(x_j-x_i)], \] (1)

where \( \varepsilon_i, \mu_i, \nu_i, \) and \( n_i \) are constants; \( \alpha_{ij}, \beta_{ij}, \delta_{ij}, \) and \( \gamma_{ij} \) are the inductive, resistive, capacitive, and diffusive coupling factors, respectively; and \( i, j = 1, 2, 3, \ldots, k. \)

We restrict the consideration to the case of two resistively coupled subsystems \((i, j = 1, 2)\) with \( n_1 = m \) and \( n_2 = n \) and introduce signal \( f(t) \) in each of the subsystems. Then, we have

\[ d^2 x_1/dt^2 - \varepsilon_1(1-x_1^2)dx_1/dt + \mu_1x_1 + v_1x_1^m = \beta_{12} dx_2/dt + f(t), \]

\[ d^2 x_2/dt^2 - \varepsilon_2(1-x_2^2)dx_2/dt + \mu_2x_2 + v_2x_2^n = \beta_{21} dx_1/dt + f(t). \] (2)

It follows from [4–6] that chaotic oscillations are excited in the system described by Eqs. (2) if \( f(t) \) is defined as a function depending on the natural oscillations of the system. For example, this function may be defined by the condition

\[ f(t) = \begin{cases} az & \text{at } x_1 > x_2 \\ bu & \text{at } x_2 > x_1. \end{cases} \] (3)
condition) rather than the external action. Thus, chaos emerges owing to self-chaotization of the self-oscillations in the system.

When the PCA provides for chaotization of oscillations, the following system of equations should be employed instead of system (2) in the case of resistive coupling:

\[
\begin{align*}
    d^2x_1/dt^2 &- \varepsilon_1(1 + f(t)x_1 - x_1^2)dx_1/dt \\
    + \mu_1x_1 + v_1x_1^m = \beta_{12}dx_2/dt, \\
    d^2x_2/dt^2 &- \varepsilon_2(1 + f(t)x_2 - x_2^2)dx_2/dt \\
    + \mu_2x_2 + v_2x_2^m = \beta_{21}dx_1/dt.
\end{align*}
\]

Equations (3)–(5) describe a mathematical model of a PDA self-oscillating system that also exhibits chaotic dynamics.

If \( f(t) = 0 \), Eqs. (2) and (5) describe a system of two mutually coupled van der Pol–Duffing oscillators. When \( f(t) = 0 \) and the system has unidirectional coupling (e.g., at \( \beta_{21} = 0 \)), these equations describe forced synchronization of the oscillations of the first oscillator by the oscillations of the second oscillator. If \( \beta_{12} = \beta_{21} = 0 \), \( m = n = 3 \), and \( f(t) = A_s \cos \omega t \) (\( A_s \) and \( \omega \) are the amplitude and frequency of the external signal, respectively), each of the equations of system (2) describes the effect of a harmonic signal studied in [1, 3].

It is expedient to analyze independently the van der Pol–Duffing equations in the presence of unidirectional and mutual coupling at \( f(t) = 0 \). The study of coupled systems is needed to illustrate the effect of the CFA and PCA in the cases of synchronous and asynchronous interactions. Therefore, we start with the particular cases (regarded as initial modes) of interaction of the subsystems.

The constants in Eqs. (2)–(5) are chosen so as to simplify the numerical analysis. Unless specified otherwise, the constants are as follows: \( \varepsilon_1 = \varepsilon_2 = 0.4, \mu_1 = \mu_2 = 0, v_1 = 1, m = 3, n = 5, \omega_z = 1.8, \omega_u = 1.6, Q_z = Q_u = 100, \delta_z = 0.44, \) and \( \delta_u = 0.36 \). Parameters \( a, b, \beta_{12}, \beta_{21} \), and \( v_2 \) are varied.

### 3. UNIDIRECTIONALLY COUPLED OSCILLATORS

The operating modes allowing for chaos in the unidirectionally coupled van der Pol–Duffing equations can be determined from the bifurcation diagrams that show the maximum value of the oscillation in one subsystem (e.g., the first subsystem) as a function of the parameter that governs variation in the self-oscillation frequency of the second subsystem. The diagrams presented in Figs. 1 and 2 serve as illustrations. These diagrams demonstrate the maximum value of oscillation \( x_1(t) \) (denoted as \( [x_1] \)) as a function of parameter \( v_2 \) for

![Fig. 1. Maximum values of oscillation \( x_1(t) \) vs. parameter \( v_2 \) at \( \beta_{12} = (a) 0.3 \) and (b) 0.6. (c) Plots of \( (I) \) the frequency and (2) the amplitude of autonomous oscillations in the second subsystem vs. parameter \( v_2 \).](image-url)