INTRODUCTION

Modern financial mathematics widely uses Brownian motion (Wiener random process) and its discrete analogue, random walk, for description of evolution of prices on financial markets [1]. Random oscillations of exchange prices are conventionally explained by the influence on the pricing process of multiple exogenous factors subject to random time variations.

In [2] a new “strategic” motivation of the appearance of random price fluctuations on financial markets is proposed. It is assumed that the market participants have different information concerning events which influence prices. Stockbrokers, who possess additional “insider” information in the case of long-term interaction, inevitably reveal this information to other market participants via their actions. However, the “insider” is not interested in disclosing his/her private information, which results in the loss of strategic advantage. The tendency of the “insider” to hide his/her information forces him/her to strategic maneuvering manifested in randomization of his/her actions. The idea of [2] is that it is this randomization that results in smoothing sharp jumps of market prices and results in the appearance of the Brownian component in their evolution.

In [2] this idea is demonstrated using the example of the simplified model of multistep exchange bids with asymmetric information. In this model two players with the opposite interests trade for “risky” assets (stocks) of the same type. Before the beginning of the bidding, the true stock price is determined by the random choice for the whole period of auction which is held in $n$ rounds. Two variants of the stock price are possible, the high and the low (zero) prices. The probability $p$ of choosing a high price is known to both players. Player 1, the insider, knows about the outcome of the random move, Player 2 does not have this information but knows about the awareness of Player 1.

At each round of bidding the players simultaneously post the bids.

The player who assigned a higher stake purchases one stock from his/her opponent for this price. Both players tend to maximize the cost of their final portfolio (money plus true price of purchased stocks). The described model is reduced to the zero-sum repeated game with incomplete information of the second player [3]. It is shown that the sequence of values $V_{n}(p)$ (maximal guaranteed insider’s payoffs) for $n$-step games with arbitrary admissible bids grows unboundedly when $n \to \infty$. The authors of [2] study the asymptotic behavior of the random sequence of transactions generated by optimal strategies of the players. They demonstrate the presence of the Brownian component in this asymptotic behavior and consider this phenomenon as the key issue for motivation of the endogenous origin of Brownian motion in finance theory.

In model [2] the players can make arbitrary bids. An unbounded growth of insider’s payoff means that in this model he/she can cheat the opponent during an arbitrarily long time. Since exchange bids go in some monetary units, it is more realistic to assume that just discrete bids multiple of the minimal monetary unit are admissible. In [4, 5] the discrete variant of the model with a high stock price equal to an integer $m > 0$ and integer admissible bids was analyzed. In this game only bids $0, 1, \ldots, m - 1$ are meaningful.

It was proved in [4] that when $n \to \infty$ the sequence of values $V_{n}(p)$ of the corresponding $n$-step games $G_{n}(p)$ is bounded and converges. The boundedness of insider’s payoffs means that in this model he/she cannot cheat the opponent for an arbitrarily long time, and the opponent reveals the true stock cost in the
course of the game. Therefore, it is reasonable to consider games $G^n_m(p)$ with the number of steps not limited a priori. We showed that the value $V^n_m(p)$ of this game is equal to the limit $V^m_n(p)$ and both players have optimal strategies. The dependence $V^m_n(p)$ represents a continuous concave piecewise linear function with $m$ linearity domains $[k/m, (k + 1)/m]$, $k = 0, \ldots, m - 1$, well defined by its values at the non-smoothness points $V^m_n(k/m) = k(m - k)/2$. In particular, the function $V^3_n(p)$ has two non-smoothness points $1/3$ and $2/3$ with the values $V^3_n(1/3) = V^3_n(2/3) = 1$.

The optimal strategy of Player 1 generates an elementary random walk of posterior expected stock prices over the points $0, 1, \ldots, m$ with absorption at the extreme points. Absorption means revealing the true stock price by Player 2. The game value $V^m_n(k/m)$ is equal to the expected number of steps till absorption $k(m - k)$ multiplied by the constant one-step payoff of Player 1 equal to $1/2$. However, the problem of solution for the $n$-step game $G^n_m(p)$ with discrete admissible bids still remains open. The case of two admissible bids ($m = 2$) is trivial, the true stock price is revealed by Player 2 at the first step, and $V^2_n(p) = V^2_n(p) = \min\{p, 1 - p\}$.

In the case of three admissible bids ($m = 3$), the situation becomes qualitatively more complicated. In this case the solution even for the one-step game $G^3_n(p)$ is nontrivial.

In this paper we obtain in an explicit form the solutions for the $n$-step game $G^n_m(p)$ with three admissible bids. The optimal strategies of the players are expressed using the second-order recursive sequence $\delta_n$ determined by the recurrence relations $\delta_{n+1} = 2(\delta_n + \delta_{n-1})$. It is shown that the piecewise linear continuous concave value function $V^3_n(p)$ of the game $G^3_n(p)$ on the interval $[0, 1]$ has three non-smoothness points, $1/3$, $p_n \in (1/3, 2/3)$, and $2/3$. The values of the function $V^3_n(p)$ at these points are also determined using the recursive sequence $\delta_n$. When $n \to \infty$ the sequences of values $V^3_n(1/3)$, $V^3_n(p_n)$, and $V^3_n(2/3)$ converge to unity. Thus, in the limit, the non-smoothness point $p_n$ disappears and the functions $V^3_n(p)$ converge to the value $V^3_n(p)$ of the game with unbounded duration $G^3_n(p)$ [4].

1. STATEMENT OF THE PROBLEM.

MULTISTAGE BIDDING WITH THREE ADMISSIBLE BIDS

In the considered model two exchange players possess money and single-type stocks. The true stock price is determined by the “state of nature” $s \in \{L, H\}$ and is equal to three in the state $H$ and to zero in the state $L$. Before the bidding starts, the random move chooses the “state of nature” $H$ with the probability $p$ or $L$ with the probability $1 - p$. Both players know the probability $p$. The “state of nature” is reported to Player 1 and is not reported to Player 2. Player 2 knows of the awareness of Player 1.

Then the players participate in the multistage bidding. At each bidding step $t = 1, 2, \ldots, n$ the players independently and simultaneously post bids, $i_t$ is the bid of Player 1 and $j_t$ is the bid of Player 2. In the considered model any integer bids are admissible, but just three bids are meaningful: $0, 1, 2$. Note that the stake equal to 3 (high stock price) is inefficient. If $i_t > j_t$, Player 1 purchases for the price $i_t$ one stock from the opponent. If $i_t < j_t$, Player 2 purchases for the price $j_t$, one stock from the opponent. If $i_t = j_t$, nothing happens. The players tend to maximize the price of their final portfolio.

This model is naturally described by the zero-sum repeated game $G^3_n(p)$ with incomplete information of Player 2, the state space $S = \{L, H\}$, and the sets of actions of the players $I = J = \{0, 1, 2\}$. In the state $L$ one-step payoffs of Player 1 are determined by the following matrix: $A^L = \{a^L_{ij}\}, i, j \in J,$

$$
\begin{bmatrix}
0 & 1 & 2 \\
-1 & 0 & 2 \\
-2 & -2 & 0
\end{bmatrix}
$$

Correspondingly, in the state $H$ one-step payoffs of Player 1 are determined by the following matrix: $A^H = \{a^H_{ij}\}, i, j \in J,$

$$
\begin{bmatrix}
0 & -2 & -1 \\
2 & 0 & -1 \\
1 & 1 & 0
\end{bmatrix}
$$

At the end of the game, Player 2 pays to Player 1 the sum of one-step payoffs,

$$
\sum_{t=1}^n a'(i_t, j_t),
$$

where $s \in \{L, H\}$ is the element randomly chosen at the zero step.

It is known (see, e.g., [3]) that both players can forget preceding actions of uninformed Player 2. Therefore, at step $t$ it is sufficient for both players to take into account just the sequence $(i_0, \ldots, i_{t-1})$ of preceding actions of Player 1. The actions of informed Player 1,