Initial–Boundary Value Problem for the Volterra Lattice on a Half-Line with Zero Boundary Condition

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The Volterra lattice, which is an infinite system of nonlinear differential equations, finds important applications in plasma physics and zoology (see [1–4]). It is well known that the Cauchy problem for the Volterra lattice with converging initial data can be analyzed in detail by the inverse scattering method (ISM) (see [1, 4]). Also of interest are problems with initial data having more complex behavior at infinity (see [4]), which have received little attention thus far.

For sequences of positive functions \( a_n(t) \in C^1 (-\infty, \infty) \), consider the following initial–boundary value problem for the Volterra lattice:

\[
\begin{align*}
\dot{a}_n(t) &= a_n(t)[a_{n-1}(t) - a_{n+1}(t)], \quad n = 1, 2, \ldots, \quad \cdot = \frac{d}{dt}, \\
a_n(0) &= a_n, \quad n = 1, 2, \ldots, \\
a_0(t) &= 0,
\end{align*}
\]

where

\[
\sum_{n=1}^{\infty} n |a_n - \hat{a}_n| < \infty,
\]

in which \( a_n \) satisfies the condition

\[
\sum_{n=1}^{\infty} n |a_n - \hat{a}_n| < \infty,
\]

where \( \hat{a}_{n+N} = \hat{a}_n > 0 \) for \( n = 1, 2, \ldots \) and \( N \) is a positive integer. We search for a solution \( a_n(t) \) to problem (1)–(3) such that \( a_n(t) - \hat{a}_n \) is a rapidly decreasing function; i.e., for any \( T > 0 \) it satisfies

\[
\| Q_1(t) \|_{C^1[-T, T]} < \infty,
\]

where

\[
Q_1(t) = \sum_{n=1}^{\infty} n' |a_n(t) - \hat{a}_n|; \text{ here, } r = 1 \text{ (or } r = 3).\]

In this paper, the ISM is used to analyze the existence of a solution to problem (1)–(3) in the class

\[
\| Q_1(t) \|_{C^1[-T, T]} < \infty. \]

A similar task concerning the Cauchy problem and the initial–boundary value problem for various nonlinear equations was addressed in [3, 5–8]. However, the application of the ISM to the initial–boundary value problem is much more complicated than to the Cauchy problem (see [7] and the references therein). Hence, this issue is of special interest.

Note that problem (1)–(3) with bounded initial data was considered in [3]. The existence of its solution in the class of sequences \( a_n(t) \) that are locally uniformly bounded with respect to \( t \) was also established in [3]. However, the method proposed in [3] fails to analyze the solvability of problem (1)–(3) in the class

\[
\| Q_1(t) \|_{C^1[-T, T]} < \infty. \]

The affirmative answer to this question is given by the following result.

**Theorem 1.** Problem (1)–(3) has a unique solution in the class \( \| Q_1(t) \|_{C^1[-T, T]} < \infty \) if \( N \leq 2 \) and \( Q_1(0) < \infty \).

It turns out that the solvability of problem (1)–(3) is closely related to the period of the sequence \( \hat{a}_n \).

**Theorem 2.** Problem (1)–(3) has no solutions in the class \( \| Q_1(t) \|_{C^1[-T, T]} < \infty \) if the number \( N > 2 \) is the least period of the sequence \( \hat{a}_n \).

1. To study problem (1)–(3), we use the inverse scattering theory constructed in [9, 10] for the boundary problem

\[
\begin{align*}
a_{n-1}y_{n-1} + a_{n+1}y_{n+1} &= \lambda y_n, \quad n \geq 1, \quad a_0 = \hat{a}_2, \\
y_0 &= 0,
\end{align*}
\]

where \( N = 2 \) and \( Q_1(0) < \infty \). Problem (4), (5) has the continuous spectrum \( \sigma = \{ \lambda: (\hat{a}_1 - \hat{a}_2)^2 \leq \lambda^2 \leq (\hat{a}_1 + \hat{a}_2)^2 \} \) and eigenvalues \( \pm \nu_k \) (with \( \nu_k \geq 0 \) for \( k = 1, 2, \ldots, p \)) lying outside \( \sigma \).

The coefficient \( a_n \) is extended by setting \( a_{-1} = a_0 = \hat{a}_2 \). In addition (4), consider the equation

\[
a_{n-1}y_{n-1} + a_{n+1}y_{n+1} = \lambda y_n, \quad n \geq 0, \quad a_{-1} = a_0 = \hat{a}_2.\]
Let $\Gamma$ denote the complex $\lambda$-plane with cuts along $\sigma$. In $\Gamma$ consider the function
\[ z = z(\lambda) = \frac{\lambda^2 - \bar{a}_1^2 - \bar{a}_2^2}{2\bar{a}_1\bar{a}_2} + \sqrt{\frac{(\lambda^2 - \bar{a}_1^2 - \bar{a}_2^2)^2 - 1}{2\bar{a}_1\bar{a}_2}}, \]
where the regular branch is determined by the condition
\[ \sqrt{\frac{(\lambda^2 - \bar{a}_1^2 - \bar{a}_2^2)^2 - 1}{2\bar{a}_1\bar{a}_2}} < 0 \text{ for } \lambda > \bar{a}_1 + \bar{a}_2. \]
Let $e_{2n} = \hat{a}_2^*z + \frac{\bar{a}_1}{\lambda} z^n$ and $e_{2n-1} = z^n$ for $n = 0, 1, 2, \ldots$. It follows from [9, 10] that, if $N = 2$ and $Q_1(0) < \infty$, Eq. (4) has a solution $f_n(\lambda)$ that is representable in the form
\[ f_n(\lambda) = \alpha_n \left( e_n + \sum_{m=1}^{\infty} K(n, n + 2m) e_{n+2m} \right), \quad n \geq -1. \]
Moreover, we have
\[ \frac{a_{n-1}}{a_n} = \frac{\alpha_n}{\alpha_{n-1}}, \]
\[ \frac{a_n^2}{a_n} = \hat{a}_n + \hat{a}_{n-1} K(n, n + 2) - \hat{a}_{n-1} K(n - 1, n + 1), \]
where $\hat{a}_0 = \hat{a}_2$.

Define
\[ S(\lambda) = \frac{f_0(\lambda)}{f_0(\lambda)}, \quad u_n(\lambda) = f_n(\lambda) - S(\lambda) f_n(\lambda), \quad \lambda \in \partial \Gamma, \]
\[ M_k^2 = \sum_{n=1}^{\infty} f_n^2(\nu_k), \quad u_n(\pm \nu_k) = M_k f_n(\pm \nu_k), \quad k = 1, 2, \ldots, p. \]
Then, for $\lambda \in \partial \Gamma$ and $\lambda = \pm \nu_k$, the vectors $\{u_n(\lambda)\}_{n=1}^{\infty}$ form the complete set of generalized eigenfunctions of problem (4), (5):
\[ \sum_{\lambda = \pm \nu_k} u_n(\lambda) u_m(\lambda) + \frac{1}{2\pi \partial \Gamma} \int \frac{\lambda}{\bar{a}_1\bar{a}_2(z - \bar{z}^{-1})} u_n(\lambda) u_m(\lambda) d\lambda = \delta_{nm}, \]
where $\delta_{nm}$ is the Kronecker delta.

The set $\{S(\lambda); \nu_k; M_k, k = 1, 2, \ldots, p\}$ is called the scattering data for boundary problem (4), (5). In [9, 10], for $Q_1(0) < \infty$, characteristic properties of the scattering data were established that make it possible to uniquely reconstruct the coefficient $a_n$ from the scattering data. Specifically, let
\[ F(n, m) = \sum_{\lambda = \pm \nu_k} M_k^2 e_n(\lambda) e_m(\lambda) + \frac{1}{2\pi \partial \Gamma} \int \frac{\lambda S(\lambda)}{\bar{a}_1\bar{a}_2(z - \bar{z}^{-1})} e_n(\lambda) e_m(\lambda) d\lambda. \]

Then the coefficient $a_n (n \geq 1)$ is recovered by formula (7), where $K(n, n + 2m)$ solves the Marchenko-type basic equation
\[ F(n, n + 2m) + K(n, n + 2m) + \sum_{r=1}^{\infty} K(n, n + 2r) F(n + 2r, n + 2m) = 0, \quad n \geq -1, \quad m \geq 1 \]
and $\alpha_n > 0$ is determined by the equality
\[ \alpha_n^2 = 1 + F(n, n) + \sum_{r=1}^{\infty} K(n, n + 2r) F(n + 2r, n), \quad n \geq 1. \]

2. Now let the coefficient $a_n(t), (n \geq 1)$ in Eqs. (4) and (4') be a solution to problem (1)–(3) in the class $||Q(t)||_{L^2[T, T]} < \infty$. Then the evolution of the scattering data is described [9] by the formulas
\[ S(\lambda, t) = S(\lambda, 0) \exp\{\hat{a}_1\hat{a}_2(z^{-1} - z)t\}, \quad v_k(t) = v_k(0), \quad k = 1, 2, \ldots, p. \]

To find the solution to problem (1)–(3) at $N = 2$, we have to determine the scattering data at $t = 0$ and, then, to construct $F(n, m, t)$ by formula (9), where $S(\lambda, 0)$ and $M_k(0)$ are given by (12). Next, an equation of type (10) with a parameter $t$ is solved for $K(n, m, t)$ and $a_n(t)$ is determined by the formula
\[ \frac{a_n^2(t)}{\hat{a}_n} = \hat{a}_n + \hat{a}_{n+1} K(n, n + 2, t) - \hat{a}_{n-1} K(n - 1, n + 1, t), \quad n \geq 1. \]

3. The proof sketch of Theorem 1 is as follows. Let us show that the function $a_n(t)$ constructed as described above satisfies (1) and (3).