The Volterra lattice, which is an infinite system of nonlinear differential equations, finds important applications in plasma physics and zoology (see [1–4]). It is well known that the Cauchy problem for the Volterra lattice with converging initial data can be analyzed in detail by the inverse scattering method (ISM) (see [1, 4]). Also of interest are problems with initial data having more complex behavior at infinity (see [4]), which have received little attention thus far.

For sequences of positive functions \(a_n(t) \in C^{(1)}(\infty, \infty)\), consider the following initial–boundary value problem for the Volterra lattice:

\[
\dot{a}_n(t) = \frac{1}{2} a_n(t) \left[ a_{n-1}^2(t) - a_{n+1}^2(t) \right], \quad n = 1, 2, \ldots, \quad \dot{n} = \frac{d}{dt},
\]

\[
a_n(0) = a_n, \quad n = 1, 2, \ldots,
\]

\[
a_0(t) = 0,
\]

in which \(a_n\) satisfies the condition \(\sum_{n=1}^{\infty} |a_n - \hat{a}_n| < \infty\), where \(\hat{a}_{n+N} = \hat{a}_n > 0\) for \(n = 1, 2, \ldots\) and \(N\) is a positive integer. We search for a solution \(a_n(t)\) to problem (1)–(3) such that \(a_n(t) - \hat{a}_n\) is a rapidly decreasing function; i.e., for any \(T > 0\), it satisfies

\[
\|Q_1(t)\|_{C([-T, T])} < \infty,
\]

where \(Q_1(t) = \sum_{n=1}^{\infty} n |a_n(t) - \hat{a}_n|\); here, \(r = 1\) (or \(r = 3\)).

In this paper, the ISM is used to analyze the existence of a solution to problem (1)–(3) in the class

\[
\|Q_1(t)\|_{C([-T, T])} < \infty, \quad \|Q_1(t)\|_{C([-T, T])} < \infty.
\]

A similar task concerning the Cauchy problem and the initial–boundary value problem for various nonlinear equations was addressed in [3, 5–8]. However, the application of the ISM to the initial–boundary value problem is much more complicated than to the Cauchy problem (see [7] and the references therein). Hence, this issue is of special interest.

Note that problem (1)–(3) with bounded initial data was considered in [3]. The existence of its solution in the class of sequences \(a_n(t)\) that are locally uniformly bounded with respect to \(t\) was also established in [3]. However, the method proposed in [3] fails to analyze the solvability of problem (1)–(3) in the class \(\|Q_1(t)\|_{C([-T, T])} < \infty\). The affirmative answer to this question is given by the following result.

**Theorem 1.** Problem (1)–(3) has a unique solution in the class \(\|Q_1(t)\|_{C([-T, T])} < \infty\) if \(N \leq 2\) and \(Q_1(0) < \infty\). It turns out that the solvability of problem (1)–(3) is closely related to the period of the sequence \(\hat{a}_n\).

**Theorem 2.** Problem (1)–(3) has no solutions in the class \(\|Q_1(t)\|_{C([-T, T])} < \infty\) if the number \(N > 2\) is the least period of the sequence \(\hat{a}_n\).

To study problem (1)–(3), we use the inverse scattering theory constructed in [9, 10] for the boundary problem

\[
a_{n-1}y_{n-1} + a_ny_{n+1} = \lambda y_n, \quad n \geq 1, \quad a_0 = \hat{a}_2, \quad (4)
\]

\[
y_0 = 0, \quad (5)
\]

where \(N = 2\) and \(Q_1(0) < \infty\). Problem (4), (5) has the continuous spectrum \(\sigma = \{ \lambda; (\hat{a}_1 - \hat{a}_2)^2 \leq \lambda^2 \leq (\hat{a}_1 + \hat{a}_2)^2 \}\) and eigenvalues \(\pm v_k\) (with \(v_k \geq 0\) for \(k = 1, 2, \ldots, p\)) lying outside \(\sigma\).

The coefficient \(a_n\) is extended by setting \(a_{-1} = a_0 = \hat{a}_2\). In addition (4), consider the equation

\[
a_{n-1}y_{n-1} + a_ny_{n+1} = \lambda y_n, \quad n \geq 0, \quad a_{-1} = a_0 = \hat{a}_2. (4')
\]
Let $\Gamma$ denote the complex $\lambda$-plane with cuts along $\sigma$. In $\Gamma$ consider the function

$$z = z(\lambda) = \frac{\lambda^2 - \hat{a}_1^2 - \hat{a}_2^2}{2\hat{a}_1\hat{a}_2} + \frac{\left(\lambda^2 - \hat{a}_1^2 - \hat{a}_2^2\right)^2}{2\hat{a}_1\hat{a}_2} - 1,$$

where the regular branch is determined by the condition

$$\frac{\left(\lambda^2 - \hat{a}_1^2 - \hat{a}_2^2\right)^2}{2\hat{a}_1\hat{a}_2} < 0$$

for $\lambda > \hat{a}_1 + \hat{a}_2$. Let $e_{2n} = (\hat{a}_2z + \frac{1}{\lambda}z^n)$ and $e_{2n-1} = z^n$ for $n = 0, 1, 2, \ldots$. It follows from [9, 10] that, if $N = 2$ and $Q_2(0) < \infty$, Eq. (4') has a solution $f_n(\lambda)$ that is representable in the form

$$f_n(\lambda) = \alpha_n \left( e_n + \sum_{m=1}^{\infty} K(n, n + 2m) e_{n + 2m} \right),$$

$$n \geq -1.$$ 

Moreover, we have

$$\frac{\alpha_{n-1}}{\alpha_n} = \frac{a_{n-1}}{a_n},$$

$$\frac{a_n^2}{a_n} = \hat{a}_n + \hat{a}_{n-1}K(n, n + 2) - \hat{a}_{n-1}K(n - 1, n + 1),$$

where $\hat{a}_0 = \hat{a}_2$.

Define

$$S(\lambda) = \frac{f_0(\lambda)}{f_0(\lambda)}, \quad u_n(\lambda) = f_n(\lambda) - S(\lambda)f_n(\lambda),$$

$$\lambda \in \partial\Gamma,$$

$$M_k^2 = \sum_{n=1}^{\infty} f_n^2(v_k), \quad u_n(\pm v_k) = M_k f_n(\pm v_k),$$

$$k = 1, 2, \ldots, p.$$ 

Then, for $\lambda \in \partial\Gamma$ and $\lambda = \pm v_k$, the vectors $\{u_n(\lambda)\}_{n=1}^{\infty}$ form the complete set of generalized eigenfunctions of problem (4), (5):

$$\sum_{\lambda = \pm v_k} u_n(\lambda)u_m(\lambda) + \frac{1}{2\pi i} \int_{\partial\Gamma} \frac{\lambda}{\hat{a}_1\hat{a}_2(z - z^{-1})} u_n(\lambda)u_m(\lambda) d\lambda = \delta_{nm},$$

where $\delta_{nm}$ is the Kronecker delta.

The set $\{S(\lambda); v_k; M_k, k = 1, 2, \ldots, p\}$ is called the scattering data for boundary problem (4), (5). In [9, 10], for $Q_2(0) < \infty$, characteristic properties of the scattering data were established that make it possible to uniquely reconstruct the coefficient $a_n$ from the scattering data. Specifically, let

$$F(n, m) = \sum_{\lambda = \pm v_k} M_k^2 e_n(\lambda)e_m(\lambda)$$

$$+ \frac{1}{2\pi i} \int_{\partial\Gamma} \frac{\lambda S(\lambda)}{\hat{a}_1\hat{a}_2(z - z^{-1})} e_n(\lambda)e_m(\lambda) d\lambda.$$ 

Then the coefficient $a_n$ ($n \geq 1$) is recovered by formula (7), where $K(n, n + 2m)$ solves the Marchenko-type basic equation

$$F(n, n + 2m) + K(n, n + 2m)$$

$$+ \sum_{r=1}^{\infty} K(n, n + 2r)F(n + 2r, n + 2m) = 0,$$

$$n \geq -1, \quad m \geq 1$$

and $\alpha_n > 0$ is determined by the equality

$$\alpha_n^2 = 1 + F(n, n) + \sum_{r=1}^{\infty} K(n, n + 2r)F(n + 2r, n),$$

$$n \geq 1.$$ 

2. Now let the coefficient $a_n(t)$, $(n \geq 1)$ in Eqs. (4) and (4') be a solution to problem (1)–(3) in the class $\|Q_2(t)\|_{L^r[\tau, \tau]} < \infty$. Then the evolution of the scattering data is described [9] by the formulas

$$S(\lambda, t) = S(\lambda, 0)\exp\{\hat{a}_1\hat{a}_2(z^{-1} - z)t\},$$

$$v_k(t) = v_k(0) = v_k;$$

$$M_k^2(t) = M_k^2(0)\exp\{\hat{a}_1\hat{a}_2(\mp v_k)t\},$$

$$k = 1, 2, \ldots, p.$$ 

To find the solution to problem (1)–(3) at $N = 2$, we have to determine the scattering data at $t = 0$ and, then, to construct $F(n, m, t)$ by formula (9), where $S(\lambda, 0)$ and $M_k(0)$ are given by (12). Next, an equation of type (10) with a parameter $t$ is solved for $K(n, m, t)$ and $a_n(t)$ is determined by the formula

$$\frac{a_n^2(t)}{a_n} = \hat{a}_n + \hat{a}_{n+1}K(n, n + 2, t)$$

$$- \hat{a}_{n-1}K(n - 1, n + 1, t), \quad n \geq 1.$$ 

3. The proof sketch of Theorem 1 is as follows. Let us show that the function $a_n(t)$ constructed as described above satisfies (1) and (3).