Singular Solitons and Indefinite Metrics
P. G. Grinevich$^a$ and Academician S. P. Novikov$^b$

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This paper is a continuation of [1]. Consider the class of potentials $u(x)$ with $x \in \mathbb{R}$ which are $C^\omega$-smooth outside isolated singularities $x_j$. In a neighborhood of the points $x_j$, the potentials must be meromorphic functions in a small neighborhood of $x_j \in \mathbb{C}$. Let $u(x)$ be real.

**Proposition 1.** All solutions $\Psi(x, \lambda)$, $\lambda \in \mathbb{C}$ of the equation $L\Psi = -\Psi'' + u(x)\Psi = \lambda\Psi$, where $\lambda \in \mathbb{C}$, are meromorphic functions of $x$ in a neighborhood of the points $x_j$ if and only if the following condition holds:

$$u(x) = \frac{n(n+1)k^2}{x^2}, \quad u(x) = n(n+1)\sin^2(kx).$$

The last potential is called the Lamé potential. Its Dirichlet spectrum in the interval $[0, T]$ was studied by Hermite in the case where the lattice of periods $[T = 2\omega_1, 2\omega_2]$ is rectangular. On the entire line $\mathbb{R}$, this operator has no good spectral theory in the Hilbert space $L^2(\mathbb{R})$. The other potentials are Lamé degeneracies.

**Problem.** Construct a spectral theory of real operators with potentials satisfying condition A in the space of functions described below with indefinite inner product. Prove a completeness theorem.

We specify a set of singularities $x = (x_j, N_j), j = 1, 2, \ldots, N$. We assume that each period contains finitely many singularities if the potential is periodic; if the potential decreases as $|x| \to \infty$, then the total number of singularities is finite (let $u(x) = O(1/x^2)$).

In the periodic case, we also fix a multiplier (quasimomentum) $\Psi(x + T) = \kappa \Psi(x)$, where $|\kappa| = 1$.

Admissible functions must be infinitely differentiable outside singularities. Moreover, we consider the space $\mathcal{F}(\chi) \ni \Psi$ of $C^\omega$-smooth functions such that

$$\Psi(y) + (-1)^{n+1}\Psi(-y) = O(y^{n+1}), \quad y = x - x_j, \quad j = 1, 2, \ldots, N.$$  \hspace{1cm} (2)

The entire space $\mathcal{F}(\chi) \ni \mathcal{F}(\chi)$ consists of functions $\Psi$ with singularities $\phi_j$ at the points $x_j$ of the form

$$\phi_j(y) = \sum_{k=0}^{N_j} a_{jk}y^{n_j-2k}, \quad y = x - x_j. \hspace{1cm} (3)$$

This means that $\Psi - \phi_j \in \mathcal{F}(\chi)$ locally near the point $x_j$.

The inner product $\langle \Psi_1, \Psi_2 \rangle$ is defined in terms of the regularization of windings about poles in $\mathbb{C}$:

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$^a$ Landau Institute for Theoretical Physics, Russian Academy of Sciences, ul. Kosygina 2, Moscow, 117940 Russia
e-mail: pgg@landau.ac.ru

$^b$ Institute for Physical Science and Technology, University of Maryland, USA
e-mail: novikov@ipst.umd.edu
SINGULAR SOLITONS AND INDEFINITE METRICS

Fig. 1. The poles of a rational solution of the KdV equation in the complex plane.

\[ \langle \Psi_1, \Psi_2 \rangle = \int_{0}^{\tau} \Psi_1(x) \overline{\Psi_2(x)} \, dx, \quad u(x + T) = u(x), \quad (4) \]

\[ \langle \Psi_1, \Psi_2 \rangle = \int_{-\infty}^{\infty} \Psi_1(x) \overline{\Psi_2(x)} \, dx, \quad u(x) \to 0, \quad (5) \]

The product \( \Psi_1(x) \overline{\Psi_2(x)} \) has zero residue for all singularities \( x_{j} \in \mathbb{R} \); therefore, each singular point can be detoured from above or from below: the result is the same.

**Lemma 1.** This inner product is indefinite. It has precisely \( l = \sum_{j=1}^{m} l_j \), where \( l_j = \left\lfloor \frac{(n_j + 1)}{2} \right\rfloor \), negative squares for any \( \kappa \in \mathbb{S}^1 \) or for \( u \to 0 \) as \( |x| \to \infty \) (\( u(x) = O(1/x^2) \)).

Here and in what follows, brackets denote the integer part of a number.

On the space \( \mathcal{F}_{(X)}^{0} \subset \mathcal{F}_{(X)} \) of smooth functions with zeros, this inner product coincides with usual inner product and, therefore, is positive definite. It is easy to see that each coefficient \( a_{jk} \), where \( k = 0, 1, \ldots, l_j - 1 \), gives a negative square. Suppose that all points \( p_{0} \in \Gamma \) of the contour which correspond to a unimodular multiplier \( \kappa \) with \( |\kappa| = 1 \) are nonsingular on the Riemann surface \( \Gamma \).

**Theorem 1.** Each function \( \Psi \in \mathcal{F}_{(X)} \), decreasing sufficiently rapidly as \( |x| \to \infty \), decomposes into an integral and a finite sum as

\[ \Psi = \sum_{m} c_{m} \phi_{m}, \quad \lambda_{m} = \lambda_{m}(k), \quad c_{m} = \frac{\langle \Psi, \phi_{m} \rangle}{\langle \phi_{m}, \phi_{m} \rangle}. \]

Here, it is assumed that \( u(x) \) is a singular finite-gap real periodic potential and \( \Gamma \) is a Riemann surface of Bloch functions. The convergence is understood in the following sense:

(a) outside the points \( x_{j} \), the series converges in the \( C^{0} \) topology;

(b) the series for the coefficients of singular parts converges for all points \( x_{j} \) (faster than any power);

(c) let \( \tilde{\phi}_{m} \) denote the difference \( \phi_{m} - \hat{\phi}_{m} \), where \( \phi_{m} = \sum_{k=0}^{N} \frac{a_{(m)} \phi_{k}}{n_{m} - \lambda_{k}} \); then the series \( \sum_{m} c_{m} \tilde{\phi}_{m} \) converge in \( C^{0} \) for some neighborhoods of the points \( x_{j} \).

**Theorem 2.** Each function \( \Psi \in \mathcal{F}_{(X)} \), decreasing sufficiently rapidly as \( |x| \to \infty \), decomposes into an integral and a finite sum as

\[ \Psi = \int_{k \in \mathbb{R}} c_{m} \phi_{k}(x) \, dk + \sum_{m} d_{m} \phi_{m}, \]

\[ L \phi_{k} = k^{2} \phi_{k}, \quad L \phi_{m} = \lambda_{m} \phi_{m}, \]

the numbers \( \lambda_{m} \) may be complex. Here, \( u(x) = O(1/x^2) \) as \( |x| \to \infty \). It is assumed that \( u(x) \) is a singular real multisoliton potential (i.e., an algebro-geometric degenerate potential).

For eigenfunction decompositions of regular finite-gap potentials, similar results were obtained earlier in [3]. In this case, the Hilbert space is positive definite.

**Conjecture.** Theorems 1 and 2 on completeness are valid for all rapidly decreasing and periodic potentials with singularities of class described above. Possibly, they are valid for growing potentials with the same sin-