This paper studies the local geometry of weighted Carnot–Carathéodory spaces, which are a substantial generalization of classical sub-Riemannian spaces (see [1–6] and the references therein) naturally arising in models of nonholonomic and quantum mechanics, neurobiology, optimal control theory [4–7], the theory of subelliptic equations [1, 8, 9], etc. The intensive development of sub-Riemannian geometry and its applications has led to considering more general settings of problems and inventing new methods of their investigation. The main directions of generalization are reducing the smoothness of generating vector fields and relaxing the Hörmander condition on the generating of the entire tangent space by the commutators of horizontal vector fields (e.g., such a situation arises in nonlinear optimal control theory [5, 7]).

In this paper, we define weighted Carnot–Carathéodory spaces, see Definition 1 given below; such spaces include, as special cases, both classical sub-Riemannian spaces and more general spaces studied in, e.g., [8, 9], [10–14] (regular Carnot–Carathéodory spaces under the minimal smoothness assumption) [15]. As shown below, the presence of an additional formal weight structure leads to a new effect; namely, different choices of weights may lead to different combinations of regular and nonregular points in the space under consideration (see Definition 2 and Example 1). Note also that, under this approach, the intrinsic sub-Riemannian metric, which is defined in the classical case as the greatest lower bound of the lengths of all horizontal curves joining given points, may not exist. Due to the mentioned peculiarities, the development of new methods to studying such spaces is needed.

In this paper, we obtain fundamental results concerning the local geometry of the spaces under investigation, such as a theorem about the divergence of integral lines, a tangent cone theorem, and a local approximation theorem. Moreover, all estimates are obtained in the same quasimetric, which was introduced in [8] for different purposes. The proofs are essentially based on synthesizing and generalizing methods and results obtained for the case of regular points [10, 11] and methods based on an embedding of any sub-Riemannian space in a regular space [1, 2, 4, 6, 9, 15]. We also study various geometric properties of the quasimetrics under consideration. The notion of the tangent cone to a quasimetric space, which generalizes Gromov’s theory [3] for metric spaces, was introduced in [14].

Definition 1. Suppose that, on some domain $U$ in a connected $C^\infty$-smooth manifold $\mathbb{M}$, an arbitrary number of vector fields $X_1, X_2, ..., X_q \in C^2(U)$ are given, which are assigned formal degrees $\deg(X_i) = d_i$, where $1 \leq d_1 \leq d_2 \leq ... \leq d_q \leq M$. The commutator $X_f = [X_i, [..., [X_{i-1}, X_i]...]$ is assigned the degree equal to homogeneous order:

$$\deg X_f = |f|_b = d_i + d_{i-1} + ... + d_1.$$  (1)

It is assumed that $\text{span}(X_v)_{|M} \subseteq T_v \mathbb{M}$ for all $v \in U$.

There naturally arises a filtration $H_1 \subseteq H_2 \subseteq ... \subseteq H_M = T \mathbb{M}$ of the tangent bundle, which has the property

$$[H_i, H_j] \subseteq H_{i+j}, \quad \text{where} \quad H_j = \text{span} \{X_j\}_{|M}.$$  (2)

We refer to the manifold $\mathbb{M}$ endowed with the structure introduced above as a weighted Carnot–Carathéodory space.

Note that conditions (1) and (2) interrelate the natural algebraic structure determined by the commutators of the fields $X_1, X_2, ..., X_q$ with the additional formal weight structure.

Setting $X_f \in C^\infty(U)$ and $d_1 = d_2 = ... = d_q = 1$ in Definition 1, we obtain a classical sub-Riemannian space. The subbundle $H_f$ is said to be horizontal, and its commutators generate the entire tangent bundle (i.e., the Hörmander condition holds).
Definition 2. A point $u \in U$ in a Carnot–Carathéodory space is said to be regular if it has a neighborhood in which the dimensions of all $H_i$ are constant; otherwise, the point is said to be nonregular.

An important feature of the spaces introduced in Definition 1 is that different choices of weights $d_i$ may lead to different distributions of regular and nonregular points in the space. Let us illustrate this by an example.

Example 1. On the space $\mathbb{R}^3$ with coordinates $(x, y, t)$, consider the system of vector fields $\{X_1 = \partial_x, X_2 = \partial_y + y\partial_t, X_3 = \partial_t\}$, for which $[X_1, X_2] = \partial_t$.

Setting $\deg(X_i) = 1$ for $i = 1, 2, 3$, we obtain $\deg([X_1, X_2]) = 2$, $H_1 = \text{span}\{X_1, X_2, X_3\}$, and $H_2 = H_1 \cup \text{span}\{X_1, X_2\}$. In this case, the plane $\{v = 0\}$ consists of nonregular points. Indeed, for $y \neq 0$, we have $\dim(H_1) = 3$, and for $y = 0$, we have $\dim(H_1) = 2$.

Setting $\deg(X_i) = a$, $\deg(X_3) = b$, and $\deg(X_3) = a + b$, where $a \leq b$, we obtain $\deg([X_1, X_2]) = a + b$. and, therefore, $H_1 = \text{span}\{X_1\}$, $H_2 = H_a \cup \text{span}\{X_1\}$, and $H_{a+b} = H_1 \cup H_2 \cup \text{span}\{X_1, X_2, X_3\}$. In this case, all points in $\mathbb{R}^3$ are regular, because $\dim(H_2) = 1$, $\dim(H_2) = 2$, and $\dim(H_{a+b}) = 3$.

To avoid cumbersome notation, we state all results for the basic model case where $d_1 = 1$ and $d_2 = M$.

Definition 3. On $U$, we define a measuring function as

$$\rho(\nu, w) = \inf\left\{\delta > 0 \mid \text{there exists a curve } \gamma: [0, 1] \to U, \text{ such that } \gamma(0) = \nu, \gamma(1) = w, \left|\dot{\gamma}(t)\right| = \sum_{j=1}^{M} w_j X_j(\gamma(t)), \left|w_j\right| < \delta \right\}.$$  

Definition 4. A quasiisometric space $(X, d_X)$ is a topological space $X$ endowed with a quasiisometric $d_X$. A quasiisometric is defined as a mapping $d_X: X \times X \to \mathbb{R}^+$ with the following properties:

(i) $d_X(u, v) \geq 0$, and $d_X(u, v) = 0$ if and only if $u = v$;

(ii) $d_X(u, v) \leq c_X d_X(v, u)$, where $1 \leq c_X < \infty$ does not depend on $u$, $v \in X$;

(iii) $d_X(u, v) \leq Q_X d_X(u, w) + d_X(w, v)$, where $1 \leq Q_X < \infty$ does not depend on $u$, $v, w \in X$ (this is a generalized triangle inequality);

(iv) the function $d_X(u, v)$ is upper semicontinuous with respect to the first argument.

If $c_X = Q_X = 1$, then $(X, d_X)$ is a metric space.

The following assertion is proved by using a similar assertion for regular points [10] and Propositions 6 and 8 stated below.

Proposition 1. The pair $(U, \rho)$ is a quasiisometric space.

Definition 5. Among the vector fields $\{X_j\}_{|I| \leq M}$ we choose a set of basis fields $\{Y_1, Y_2, ..., Y_N\}$ with the following properties:

(i) the vector fields $Y_1, Y_2, ..., Y_N$ are linearly independent at the point $u$ (and, therefore, in some neighborhood of this point);

(ii) the sum $\sum_{i=1}^{N} \deg Y_i$ of their degrees is minimal;

(iii) the sum $\sum_{j=1}^{N} |I_j|$ of the orders of the commutators of $X_j$ corresponding to the fields $Y_j$ is minimal.

The properties of the chosen vector fields $\{Y_j\}$ imply the following assertion.

Proposition 2. For each vector field $X_i$, $X_i(\nu) = \sum_{i=1}^{N} \xi_i(\nu) Y_i(\nu)$, where $\xi_i(u) = 0$ if $\deg Y_i > |I_i|$.

On $U$ we introduce canonical coordinates $x^\nu: \mathbb{R}^N \to U$ of the second kind, which are defined as $x^\nu(x_1, ..., x_N) = \exp(x_1 Y_1) \circ \exp(x_2 Y_2) \circ ... \circ \exp(x_N Y_N)(u)$.

In the coordinates of the second kind, we define dilations on $\mathbb{R}^N$ by the formula $\delta_\nu(x_1, x_2, ..., x_N) = (x_1^\nu e^{-\deg Y_1}, x_2^\nu e^{-\deg Y_2}, ..., x_N^\nu e^{-\deg Y_N})$ for $\nu > 0$.

Definition 6. A vector field $X$ on $\mathbb{R}^N$ is homogeneous of order $s$ if $\delta_\nu^s X = e^{\nu}X$, where the action of dilations on the vector field is defined by $\delta_\nu^s X(f \circ \delta_\nu) = (Xf) \circ \delta_\nu$.

The following proposition is proved by applying the technique for calculating in coordinates of the second kind developed in [11] (this technique is based on analogues of the Campbell–Hausdorff formula) and Proposition 2.

Proposition 3. In the coordinates $\Phi^\nu$, each $C^{M+1}$-smooth vector field $X_i$ (with $|I_i| \leq M$) admits the decomposition

$$X_i^\nu(x) := (\Phi^\nu)^{-1} X_i(\Phi^\nu(x)) = \sum_{j=1}^{N} d_{i,j}(x) \frac{\partial}{\partial x_j} = \sum_{i=1}^{N} \sum_{|I| \leq M} f_{i}^{(j, \alpha)} x^\alpha + o(\|x\|^{M})$$

as $\|x\| \to 0$, where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$, $|\alpha|_{\nu} = \sum_{i=1}^{N} \alpha_i \deg Y_i$, $|\alpha| = \sum_{i=1}^{N} \alpha_i$, $f_{i}^{(j, \alpha)} \in \mathbb{R}$, and $|\alpha|_{\nu}$ is the Euclidean norm of $x$.

Corollary. Any vector field $X_i \in C^M$, where $|I_i| \leq M$, can be represented in the form

$$X_i^\nu(x) = (X_i^{(|I|)})^\nu(x) + (X_i^{(|I|+1)})^\nu(x) + ... + o(\|x\|^{M}),$$

as $\|x\| \to 0,$