On Problem of the Dynamics of a Viscoelastic Medium with Memory on an Infinite Interval

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Abstract—The existence of a weak solution of a boundary value problem for a viscoelasticity model with memory on an infinite time interval is proved. The proof relies on an approximation of the original boundary value problem by regularized ones on finite time intervals and makes use of recent results concerning the solvability of Cauchy problems for systems of ordinary differential equations in the class of regular Lagrangian flows.

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1. In this work, we prove the existence of a weak solution of a boundary value problem for a viscoelastic fluid model with memory on an infinite time interval. In \( Q = (-\infty, T] \times \Omega \), where \( T > 0 \) and \( \Omega \subset \mathbb{R}^n \) \((n = 2, 3)\) is a bounded domain with \( \partial \Omega \subset C^2 \), we consider the problem

\[
\frac{\partial v}{\partial t} - \mu_1 \text{Div} \int_{-\infty}^{t} \exp \left( \frac{s-t}{\lambda} \right) \varepsilon(v(s, z(s; t, x))) ds - \mu_0 \Delta v
\]

\[+ \sum_{i=1}^{n} \nu_i \frac{\partial v}{\partial x_i} + \nabla p = f; \]

\[
\text{div } v = 0; \quad v|_{t=0, \Omega} = 0; \quad (1)
\]

\[
z(t, x) = x + \int_{-\infty}^{t} v(s, z(s; t, x)) ds, \quad (2)
\]

Here, \( v(t, x) = (v_1, \ldots, v_n) \) and \( p(t, x) \) are the sought velocity and pressure of the medium, \( f(t, x) \) is the density of external forces, \( \varepsilon(v) \) is the strain rate tensor with components \( \varepsilon_{ij}(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \), \( \mu_0, \lambda > 0 \), \( \mu_1 \geq 0 \), and \( z(t, x) \) is the solution of the Cauchy problem for the system of ordinary differential equations. The symbol \( \text{Div} \) denotes the divergence of a matrix function, i.e., a vector with coordinates given by the divergences of the column vectors of the matrix.

Problem (1), (2) follows from the general equations of motion of a viscoelastic medium with the Jeffreys–Oldroyd rheological relation involving a substantial derivative (see [1, 2]).

The integral term in the equations of motion means that the memory of the continuous medium along the trajectory of the velocity field is taken into account. Various models with memory have been studied in numerous works (see, e.g., [3, 4]). However, the contribution of memory in their mathematical formulations was considered, as a rule, on a finite interval and for a constant value of the spatial variable \( x \) (see, e.g., [5, 6]). Accordingly, system (1) was considered without Eq. (2), so the problem simplified substantially. In the case of the trajectories of particles taken into account, but on a finite interval, problem (1), (2) was addressed in [7], where its weak solvability was proved using a regularization of the velocity field \( v \) in (2). In the regularized case, Cauchy problem (2) was shown to be solvable in the classical sense; moreover, the good properties of the solution \( v \) provided good properties of the solution \( z \) to Eq. (2). Relying on recent results concerning the solvability of the Cauchy problem in the class of regular Lagrangian flows (briefly, RLFs) (see, e.g., [8, 9]), the weak solvability of problem (1), (2) on a finite time interval can be proved without using regularization in (2) (see [10]).

Problems with memory on the half-line \((-\infty, T]\) arise in numerous viscoelasticity models (see, e.g., [3]), but, to our knowledge, there are no existence results for solutions of these problems on infinite intervals \((-\infty, T]\) with memory taking into account the trajectories of motion of medium particles.
Such an existence result is proved in this paper.

Let \( V = \{ v \in W^1_2(\Omega)^n; \forall i = 0, \text{div } v = 0 \} \) and \( H \) be the closure of \( V \) in the norm of \( L_2(\Omega)^n \). Let \( V^{-1} \) be the adjoint of \( V \). By \((\cdot,\cdot)\) we denote the inner product in the Hilbert spaces \( L_2(\Omega), H, \) and \( L_2(\Omega)^n \); which one is meant will be clear from the context.

Recall some facts concerning RLFs (see, e.g., [8, 9]).

**Definition 1.** The RLF generated by \( v \) is a function \( z(\tau,t,x), (\tau,t,x) \in [r,T] \times [r,T] \times \overline{\Omega} \), satisfying the following conditions: (i) for a.e. \( x \) and every \( t \in [r,T] \), \( r \in (\infty, T) \), the function \( \gamma(\tau) = z(\tau,t,x) \) is absolutely continuous and satisfies Eq. (2); (ii) for any \( t, \tau \in [r,T] \) and an arbitrary Lebesgue measurable set \( B \subset \overline{\Omega} \) with a Lebesgue measure \( m(B) \), it holds that \( m(z(\tau,t,x)) = m(B) \); and (iii) \( z(t_i,t_i,x_i) = z(t_{j_i},t_{j_i},x_{j_i}) \) for all \( t_i \in [r,T] \), \( i = 1,2,3 \), and a.e. \( x_i \in \overline{\Omega} \).

Here, the definition of an RLF is given in the special case of a bounded domain \( \Omega \) for a field \( v \) with \( \text{div } v = 0 \).

It is well known (see [8]) that, if \( v \in L_1(r, T; W^1_p(\Omega)^n) \), \( 1 \leq p \leq +\infty \), \( \text{div } v(t,x) = 0 \), and \( \forall i = 0, |v_i|_{L_p(\partial \Omega)} = 0 \), then there exists a unique RLF generated by \( v \). Moreover, \( z(\tau,t,x) \in W^1_1(\Omega); \frac{\partial}{\partial \tau} z(\tau,t,x) = v(t, z(\tau,t,x)) \) for \( t, \tau \in [r,T] \) and a.e. \( x \in \Omega \); and \( z(\tau, t, \overline{\Omega}) = \overline{\Omega} \).

Let \( W = W_1 \) for \( n = 2 \) and \( W = W_2 \) for \( n = 3 \), where

\[
W_1 = L_2(\infty, T; V) \\
\cap L_\infty(\infty, T; H) \cap W^1_2(\infty, T; V^{-1})
\]

\[
W_2 = L_2(\infty, T; V) \\
\cap L_\infty(\infty, T; H) \cap W^1_{4/3, loc}(\infty, T; V^{-1})
\]

**Definition 2.** Let \( f \in L_2(\infty, T; V^{-1}) \). The weak solution of problem (1), (2) is a function \( v \in W \) satisfying the identity

\[
\frac{d(v,\phi)}{dt} + \mu_1 \int_\Omega \exp \left( \frac{s-t}{\lambda} \right) \mathcal{E}(v)(s, z(s,t,x))ds, \mathcal{E}(\phi)
\]

\[
-\sum_{i=1}^n \left( v_i, v_i \frac{\partial \phi}{\partial x_i} \right) + \mu_0(\mathcal{E}(v), \mathcal{E}(\phi)) = \langle f, \phi \rangle
\]

for any \( \phi \in V \) and a.e. \( t \in (\infty, T) \). Here, \( z \) is the RLF generated by \( v \).

Below is the main result of this work.

**Theorem 1.** Let \( f \in L_2(\infty, T; V^{-1}) \). Then problem (1), (2) has at least one weak solution.

To prove the theorem, we first consider the following family of regularized problems in \( Q_m = [-m,T] \times \Omega, m \in \mathbb{N} \):

\[
\frac{\partial v^m}{\partial t} + \sum_{i=1}^n \left( v_i^m, v_i^m \frac{\partial v^m}{\partial x_i} \right) - \mu_0 \Delta v^m = -\mu_1 \text{Div} \left( \int_\Omega \exp \left( \frac{s-t}{\lambda} \right) \mathcal{E}(v^m)(s, z^m(s,t,x))ds \right) + \nabla p^m = f^m(4)
\]

\[
\text{div } v^m = 0; \quad \text{div } (v^m-m, x) = 0, \quad x \in \Omega;
\]

\[
z^m(\tau,t,x) = x + \int_\tau^t \mathcal{E}(v^m)(s, z^m(s,t,x))ds,
\]

\( x \in \Omega, \quad t, \tau \in [-m,T] \).

Here, \( f^m \) is the restriction of \( f \) to \([-m,T] \times \Omega \) and \( v^m = S_{1/m} v^m \) in (5) is a regularization (smoothing) of the field \( v^m \). The regularization operator \( S_{1/m} \) (see [2, Section 7.7]) is such that \( ||S_{1/m}||_{H \rightarrow H} \leq M ||S_{1/m}||_{L\rightarrow C(\Omega)} \leq M \) and \( \lim_{m \rightarrow +\infty} ||S_{1/m}v - \text{div } v||_H = 0, \forall v \in H \).

For any \( m \), problem (4), (5) has at least one weak solution \( v^m \in W(m) \), where \( W(m) = L_2(\infty, T; V) \cap L_\infty(\infty, T; H) \cap W^1_2(\infty, T; V^{-1}) \) (see [7]). Moreover, Cauchy problem (5) is uniquely solvable in the classical sense.

We extend \( z^m(\tau,t,x) \) from \([-m,T] \times [-m,T] \times \overline{\Omega} \) to \((\infty, \infty) \times (\infty, \infty) \times \overline{\Omega} \) by setting \( z^m(\tau,t,x) \equiv x \) for \( \tau, t \leq -m \). Additionally, \( v^m, v^m, f^m \) are extended to \((-\infty, -m] \) by zero with the same notation retained for the extensions. Note that the extended \( z^m \) is the RLF generated by the extended \( v^m \).

With the help of the scheme from [7], we can show that

\[
\sup \left| \frac{v^m(t,r)}{t} \right|_{L_\infty} + \left| \frac{v^m}{f^m} \right|_{L_2(\infty, T; V^{-1})} \leq M \left| f \right|_{L_2(\infty, T; V^{-1})},
\]

\[
\left| \frac{v^m}{f^m} \right|_{L_2(\infty, T; V^{-1})} \leq M \left| f \right|_{L_2(\infty, T; V^{-1})}^2 \forall r \in (\infty, T).
\]

Estimates (6) imply (see [11, 12]) that \( v^m \) converges (up to a subsequence) weakly in \( L_2(\infty, T; V) \) and \( * \)-weakly in \( L_\infty(\infty, T; H) \) to \( v \in L_2(\infty, T; V) \cap L_\infty(\infty, T; H) \).

Moreover, estimates (6) imply (see, e.g., [11, Chapter III, the proof of Theorem 3.2]) that \( v^m \) converges (up to a subsequence) to \( v \) a.e. on \([-k,T] \times \Omega \) for any \( k > 0 \) and, hence, on \((\infty, T) \times \Omega \).

By applying Corollaries 3.6–3.9 from [9], we obtain the following result.