Example of Blue Sky Catastrophe Accompanied by a Birth of Smale–Williams Attractor

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Abstract—A model system is proposed, which manifests a blue sky catastrophe giving rise to a hyperbolic attractor of Smale–Williams type in accordance with theory of Shilnikov and Turaev. Some essential features of the transition are demonstrated in computations, including Bernoulli-type discrete-step evolution of the angular variable, inverse square root dependence of the first return time on the bifurcation parameter, certain type of dependence of Lyapunov exponents on control parameter for the differential equations and for the Poincaré map.


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Uniformly hyperbolic strange attractors have strong chaotic properties and allow for far-reaching mathematical analysis [1–7]. They are structurally stable, i.e. insensitive in respect to variations of functions and parameters in the governing equations. Traditionally, in reviews and textbooks on nonlinear dynamics, the uniformly hyperbolic attractors are illustrated by artificially constructed discrete-time evolution rules (Plykin attractor, Smale–Williams solenoid). An interesting problem is search for examples of such attractors in systems of physical or technical origin [8, 9]. Certainly, one possible direction of thought is consideration of scenarios of appearance of the hyperbolic chaotic attractors in nonlinear dissipative systems under variation of their control parameters.1)

Possible occurrence of the hyperbolic strange attractors was discussed by Ruelle, Takens, and Newhouse in a framework of generic phenomena accompanying destruction of quasiperiodic motions on tori of dimension 3 and more [10, 11], but they did not present concrete examples. More recently, Shilnikov and Turaev [12, 13] indicated a possibility of appearance of attractors of Smale–Williams type in Poincaré section of continuous-time systems undergoing a kind of the so-called blue sky catastrophe.

In the simplest version, a blue sky catastrophe occurs in three-dimensional phase space. At the bifurcation, a saddle-node limit cycle takes place, from which the trajectories depart along the unstable manifold, and return back to the same cycle from the opposite side. With a shift of a value of the control parameter in one direction, the saddle-node cycle transforms to a pair of close limit cycles, one stable and another unstable. With a parameter shift in the opposite direction, the saddle-node cycle disappears; instead, an attracting large-scale limit cycle emerges containing helical coils near the former saddle-node cycle.

Having initial angular coordinate \( \varphi \) near the saddle-node cycle, after a travel along the unstable manifold and subsequent return, the angular coordinate is expressed by relation, which contains, in general, an additive term \( m\varphi \); for the three-dimensional case, the integer \( m \) may be equal 0 or 1. However, at higher dimensions, any integer can occur. In particular, \( m = 2 \) corresponds to a birth

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1) This matter certainly is different from the commonly known scenarios of transition to chaos (like Feigenbaum’s one), which lead to the onset of non-hyperbolic attractors.

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of hyperbolic strange attractor represented by a Smale–Williams solenoid in Poincaré section. The authors [12, 13] stress that due to such bifurcation, a system of Morse – Smale class with simple dynamical behavior, immediately turns to a system with complex dynamics associated with the structurally stable chaotic attractor.

To my knowledge, no concrete systems with the last variant of the blue sky bifurcation were described in the literature. Known examples of the blue sky catastrophes relate to the three-dimensional case, when the hyperbolic chaotic attractor cannot arise [14–16].

To construct an example with birth of the Smale–Williams attractor, one must have the phase space dimension at least equal four. The phase-space flow has to be organized in such way that a toroidal domain close initially to the saddle-node cycle, in the course of travel and return back to that cycle would form a double-folded loop, with thinner coils. To do this, an idea advanced in our previous work with Pikovsky [17] may be applied with certain modifications.

Let us start with a two-dimensional predator – prey system with an instant state specified by two non-negative variables $r_1, r_2$:

$$
\dot{r}_1 = 2(1 - r_2 + \frac{1}{2}r_1 - \frac{1}{50}r_1^2)r_1, \quad \dot{r}_2 = 2(r_1 - \mu + \frac{1}{2}r_2 - \frac{1}{50}r_2^2)r_2.
$$

These equations differ from those in Ref. [17] with additional nonlinear terms in the second (“predator”) equation, and contain a control parameter $\mu$. If the value of $\mu$ is slightly less than $\mu_0 = \frac{3}{8}$, a picture of orbits on the phase plane $r_1, r_2$ looks like that shown in Fig. 1a. There are four fixed points here, an unstable focus A, saddles B and C_1, and a node C_2. With increase of $\mu$, the fixed points C_1 and C_2 move to meet each other at $\mu = \mu_0$, and then disappear (see the panels (b) and (c), respectively). Instead of the former pair of fixed points, a domain of relatively slow motion appears there, while the attractor is a limit cycle, which passes close to the origin and to the saddle B.

Following [17], let us consider the quantities $r_1, r_2$ as squared absolute values of complex amplitudes for two oscillators of some frequency $\omega_0$, namely, $r_{1,2} = |a_{1,2}|^2$. One can write down a set of differential equations for the complex variables $a_1, a_2$, and add terms of certain form, which introduce additional coupling between the oscillators. The equations read

$$
\dot{a}_1 = -i\omega a_1 + (1 - |a_2|^2 + \frac{1}{2}|a_1|^2 - \frac{1}{50}|a_1|^4)a_1 + \frac{1}{2}\epsilon\text{Im}a_2^2,
$$

$$
\dot{a}_2 = -i\omega a_2 + (|a_1|^2 - \mu + \frac{1}{2}|a_2|^2 - \frac{1}{50}|a_2|^4)a_2 + \epsilon\text{Re}a_1,
$$

(2)

where the added terms contain the coefficient $\epsilon$. As $a_1, a_2$ are complex, in real dynamical variables $(\text{Re}a_1, \text{Im}a_1, \text{Re}a_2, \text{Im}a_2)$ this is a four-dimensional system.

At $\epsilon = 0$, equations for $r_{1,2} = |a_{1,2}|^2$ derived from (2) coincide precisely with Eqs. (1). At $\epsilon$ small enough, and at values of $\mu$ notably less than $\mu_0$, the sustained dynamics presented graphically on

Fig. 1. Phase portraits of the system (1) with increase of $\mu$ from (a) to (c). Two fixed points, a stable node C_2 and a saddle C_1 (a) meet together at the bifurcation (b), and a limit cycle appears shown with a bold curve (c)