Interaction of Order and Convexity*1

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Abstract—This is an overview of merging the techniques of vector lattice theory and convex geometry.

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Mathematics is not a divine gift. Mathematics is a human enterprise, challenge, and endeavor. The gift of mathematics goes from master to student. The alternating chain of masters and students is the true savior of mathematics. Marquise de Laplace called Euler the master of us all. It is impossible to celebrate great Euler without acclaiming his great predecessors and great descendants. Geometry is a vast treasure trove founded by Euclid and enriched by his innumerous students and followers. Alexandr Danilovich Alexandrov was one of them. He became the first and foremost Russian geometer of the twentieth century. Alexandrov contributed to mathematics under the slogan: “Retreat to Euclid,” remarking that “the pathos of contemporary mathematics is the return to Ancient Greece.” Minkowski revolutionized the theory of numbers with the aid of the synthetic geometry of convex surfaces. The ideas and techniques of the geometry of numbers comprised the fundamentals of functional analysis which was created by Banach. The pioneering studies of Alexandrov continued the efforts of Minkowski and enriched geometry with the methods of measure theory and functional analysis. Alexandrov accomplished the turnround to the ancient synthetic geometry in a much deeper and subtler sense than it is generally acknowledged today. Geometry in the large reduces in no way to overcoming the local restrictions of differential geometry which bases upon the infinitesimal methods and ideas of Newton, Leibniz, and Gauss.

The works of Alexandrov [1, 2] made tremendous progress in the theory of mixed volumes of convex figures. He proved some fundamental theorems on convex polyhedra that are celebrated alongside the theorems of Euler and Minkowski. While discovering a solution of the Weyl problem, Alexandrov suggested a new synthetic method for proving the theorems of existence. The results of this research ranked the name of Alexandrov alongside the names of Euclid and Cauchy.

Alexandrov enriched the methods of differential geometry by the tools of functional analysis and measure theory, driving mathematics to its universal status of the epoch of Euclid. The mathematics of the ancients was geometry (there were no other instances of mathematics at all). Synthesizing geometry with the remaining areas of the today’s mathematics, Alexandrov climbed to the antique ideal of the universal science incarnated in mathematics. Return to the synthetic methods of mathesis universalis was inevitable and unavoidable as well as challenging and fruitful.

1. THE MINKOWSKI DUALITY

1.1. A convex figure is a compact convex set. A convex body is a solid convex figure. The Minkowski duality identifies a convex figure $S$ in $\mathbb{R}^N$ and its support function $S(z) := \sup \{(x, z) \mid x \in S\}$ for $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons, we assume that $\mathbb{R}^N$ lies in the set $\mathcal{V}_N$ of all compact convex subsets of $\mathbb{R}^N$.  

*The text was submitted by the author in English.

1An expanded version of the talk delivered at the opening of the Russian–German Geometry Meeting dedicated to the 95th anniversary of A. D. Alexandrov, St. Petersburg, June 18–23, 2007.

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1.2. The classical concept of support function gives rise to abstract convexity which focuses on the order background of convex sets.

Let \( E \) be a complete lattice \( E \) with the adjoint top \( \top := +\infty \) and bottom \( \bot := -\infty \). Unless otherwise stated, \( E \) is usually a Kantorovich space which is a Dedekind complete vector lattice in another terminology. Assume further that \( H \) is some subset of \( E \) which is by implication a (convex) cone in \( E \), and so the bottom of \( E \) lies beyond \( H \). A subset \( U \) of \( H \) is convex relative to \( H \) or \( H \)-convex, in symbols \( U \in \mathcal{V}(H,E) \), provided that \( U \) is the \( H \)-support set \( U_p^H := \{ h \in H \mid h \leq p \} \) of some element \( p \) of \( E \).

Alongside the \( H \)-convex sets we consider the so-called \( H \)-convex elements. An element \( p \in E \) is \( H \)-convex provided that \( p = \sup_{H^p} U \); i.e., \( p \) represents the supremum of the \( H \)-support set of \( p \). The \( H \)-convex elements comprise the cone which is denoted by \( \mathcal{Cn}(H,E) \). We may omit the reference to \( H \) when \( H \) is clear from the context. It is worth noting that convex elements and sets are “glued together” by the Minkowski duality \( \varphi : p \mapsto U_p^H \). This duality enables us to study convex elements and sets simultaneously.

Since the classical results by Fenchel [3] and Hörmander [4, 7] we know that the most convenient and conventional classes of convex functions and sets are \( \mathcal{Cn}(\text{Aff}(X),\mathbb{R}^X) \) and \( \mathcal{V}(X',\mathbb{R}^X) \). Here \( X \) is a locally convex space, \( X' \) is the dual of \( X \), and \( \text{Aff}(X) \) is the space of affine functions on \( X \) (isomorphic with \( X' \times \mathbb{R} \)).

In the first case the Minkowski duality is the mapping \( f \mapsto \text{epi}(f^*) \) where
\[
 f^*(y) = \sup_{x \in X} \langle y, x \rangle - f(x)
\]
is the Young–Fenchel transform of \( f \) or the conjugate function of \( f \). In the second case we return to the classical identification of \( U \) in \( \mathcal{V}(X',\mathbb{R}^X) \) and the standard support function that uses the canonical pairing \( \langle \cdot, \cdot \rangle \) of \( X' \) and \( X \).

This idea of abstract convexity lies behind many current objects of analysis and geometry. Among them we list the “economical” sets with boundary points meeting the Pareto criterion, capacities, monotone seminorms, vario-
us classes of functions convex in some generalized sense, for instance, the Bauer convexity in Choquet theory, etc. It is curious that there are ordered vector spaces consisting of the convex elements with respect to narrow cones with finite generators. Abstract convexity is traced and reflected, for instance, in [8–11].

2. POSITIVE FUNCTIONALS OVER CONVEX OBJECTS

2.1. The Minkowski duality makes \( \mathcal{V}_N \) into a cone in the space \( C(S_{N-1}) \) of continuous functions on the Euclidean unit sphere \( S_{N-1} \), the boundary of the unit ball \( 1_N \). This yields the so-called Minkowski structure on \( \mathcal{V}_N \). Addition of the support functions of convex figures amounts to taking their algebraic sum, also called the Minkowski addition. It is worth observing that the linear span \( [\mathcal{V}_N] \) of \( \mathcal{V}_N \) is dense in \( C(S_{N-1}) \), bears a natural structure of a vector lattice, and is usually referred to as the space of convex sets. The study of this space stems from the pioneering breakthrough of Alexandrov in 1937 and the further insights of Radström [5], Hörmander [4], and Pinsker [6].

2.2. It was long ago in 1954 that Reshetnyak suggested in his PhD Thesis [12] to compare positive measures on \( S_{N-1} \) as follows:

A measure \( \mu \) linearly majorizes or dominates a measure \( \nu \) provided that to each decomposition of \( S_{N-1} \) into finitely many disjoint Borel sets \( U_1,\ldots,U_m \) there are measures \( \mu_1,\ldots,\mu_m \) with sum \( \mu \) such that every difference \( \mu_k - \nu|_{U_k} \) annihilates all restrictions to \( S_{N-1} \) of linear functionals over \( \mathbb{R}^N \). In symbols, we write \( \mu \gg_{\mathbb{R}^N} \nu \).

Reshetnyak proved that
\[
 \int_{S_{N-1}} p \, d\mu \geq \int_{S_{N-1}} p \, d\nu
\]
for each sublinear functional \( p \) on \( \mathbb{R}^N \) if \( \mu \gg_{\mathbb{R}^N} \nu \). This gave an important trick for generating positive linear functionals over various classes of convex surfaces and functions.