An Approximation Algorithm for the Minimum Two Peripatetic Salesmen Problem with Different Weight Functions

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Abstract—We present a polynomial algorithm with time complexity $O(n^5)$ and approximation ratio $4/3$ (plus some additive constant) for the minimum 2-peripatetic salesman problem in a complete $n$-vertex graph with different weight functions valued 1 and 2 (abbreviated to as 2-PSP(1,2)-min-2w). This result improves the available algorithm for this problem with approximation ratio $11/7$.

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INTRODUCTION

The Traveling Salesman Problem (TSP) is one of the most well-known and actively studied discrete optimization problems. Recently, some authors study such a natural generalization of the Traveling Salesman Problem as the $m$-Peripatetic Salesman Problem ($m$-PSP) that consists in finding some $m$ edge-disjoint Hamiltonian cycles with minimal or maximal total weight in a complete undirected graph. Particular attention is given to the 2-Peripatetic Salesman Problem (2-PSP). The problem is studied both in the case of the general or metric weight function and in the special class of graphs with the edge weights equal to 1 and 2 (Problem 2-PSP(1,2)). All above-mentioned modifications of the TSP and $m$-PSP are NP-hard, which follows from the NP-completeness of the problem of existence of two edge-disjoint Hamiltonian cycles in an undirected graph [7–9].

In this paper, we consider the minimum two peripatetic salesmen problem in a complete graph with the edge weights equal to 1 and 2. Note that, for the minimum one peripatetic salesmen problem in the case of a general weight function, there is no polynomial algorithms with guaranteed performance ratios. For the problem TSP-min in the case of a metric weight functions, there is the algorithms by N. Christofides [6] and A. I. Serdyukov [3] with the ratio $3/2$. In the case of the special class of graphs with the edge weights equal to 1 and 2, C. H. Papadimitriou and M. Yannakakis (1993) constructed an algorithm with ratio $7/6$ [10]. For this problem, P. Berman and M. Karpinski (2006) also proposed an algorithm for which they announced the performance ratio $8/7$ [5], but it was not strictly substantiated.

For the problem 2-PSP(1,2)-min, Yu. V. Glazkov, E. Kh. Gimadi, and A. N. Glebov proposed an algorithm with ratio $6/5$ in the case when the weight functions for both cycles are identical [2]. In the case of the two different weight functions (Problem 2-PSP(1,2)-min-2w), an algorithm with ratio $11/7$ is proposed in [4].

The purpose of this work is to strengthen the results of [4], namely, to construct an approximation algorithm for solving the problem 2-PSP(1,2)-min-2w with the approximation ratio $4/3$ (without any additive constant) and an estimation of the time complexity $O(n^5)$. The algorithm is based on the idea of the method of [5] consisting in construction and sequential “improvement” of a couple of edge-disjoint partial tours (sets of chains and cycles) of edges with unit weight and the subsequent closure of these
tours into the disjoint Hamiltonian cycles. Under an “improvement” of the tours we understand the local transformation that decreases either the total number of chains and cycles that form the tours or the number of the one-vertex chains (singles). In the case when the desired quality of the solution was not reached, the existence of an improving transformation is guaranteed by the technique of so-called charges of the vertices (i.e., some numbers defined in a special way for each vertex in a graph) with subsequent redistribution of these charges among the vertices.

Here, the notion of a partial tour traditionally used in the most studies on the construction of approximate algorithms for the problems of one and two salesmen is substantially modified by the inclusion into the tours of so-called \((s,q)\)-trees (a generalization of singles) along with chains and cycles.

In Section 1, we introduce the needed definitions and give a general description for the algorithm. In Section 2, we formulate the main theorem on the evaluation of accuracy of the solutions obtained by the algorithm and the time complexity of the algorithm. The technique of charge redistribution among the graph vertices is described as well. Section 3 deals with the proof of the main theorem whose most part consists of the proof of the key lemma on charge redistribution.

1. THE NOTIONS USED AND DESCRIPTION OF THE ALGORITHM

Consider a complete undirected graph \(G = (V,E), |V| = n\), and two weight functions of the edges \(w_1 : E \to \{1,2\}\) and \(w_2 : E \to \{1,2\}\). A feasible solution of the problem 2-PSP\((1,2)\)-min-2w is an arbitrary pair of edge-disjoint Hamiltonian cycles \(H_1\) and \(H_2\) in \(G\). The weight of a solution \(H_1, H_2\) is defined as the number \(w_1(H_1) + w_2(H_2)\), where

\[
w_i(H_i) = \sum_{e \in H_i} w_i(e), \quad i = 1, 2.
\]

An feasible solution \(H_1^*, H_2^*\) is called optimal if it has the minimal weight which we denote by \(OPT\).

We define the length of a chain (cycle) \(P\) in \(G\) as the number of edges in \(P\). A one-vertex chain (of length 0) is called a single, and a chain of positive length, nontrivial. A chain is called long if its length is at least 3 and short, otherwise.

Let us recursively define the class of the \((s,q)\)-trees as follows: Every one-vertex tree \(T = \{v\}\) is called the \((s,q)\)-tree of rank \(k(T) = 1\) with the sets \(S(T) = \{v\}\) of \(S\)-vertices, \(Q(T) = \emptyset\) of \(Q\)-vertices, and \(E(T) = \emptyset\) of edges, and the root \(r(T) = v\) which has an empty parent \(p(v) = 0\). If \(T_1\) is some \((s,q)\)-tree of rank \(k - 1\), \(s \in S(T_1)\) is its arbitrary \(S\)-vertex, and \(P = xyz\) is an arbitrary 3-chain vertex-disjoint with \(T_1\) then we refer to \(T\) as the \((s,q)\)-tree of rank \(k(T) = k\) if

\[
S(T) = S(T_1) \cup \{x,z\}, \quad Q(T) = Q(T_1) \cup \{y\}, \quad E(T) = E(T_1) \cup \{sx, xy, yz\},
\]

and the root \(r(T) = r(T_1)\), where \(p(y) = s\) and \(p(z) = p(x) = y\). Moreover, the \(S\)-vertices \(x\) and \(z\) we call the sons of the \(Q\)-vertex \(y\), and the chain \(P = xyz\) is canonical for \(y\) in \(T\). This recurrent definition implies that each \((s,q)\)-tree \(T\) of rank \(k\) has

\[
|S(T)| = 2k - 1, \quad |Q(T)| = k - 1, \quad |E(T)| = 3(k - 1),
\]

and the degree of every \(Q\)-vertex in \(T\) is equal to 3.

**Lemma 1.** Let \(T\) be an \((s,q)\)-tree of rank \(k\), and let \(s \in S(T)\) be an arbitrary \(S\)-vertex of \(T\). Then, removing some suitable \(k - 1\) edges from \(T\), we can partition \(T\) in time \(O(k)\) into the single \(s\) and \(k - 1\) chains of length 2 so that the set of internal vertices of the so-obtained chains will coincide with \(Q(T)\).

**Proof.** If \(s = r(T)\) then, removing from \(T\) the set of edges \(\{qp(q) : q \in Q(T)\}\), we obtain the partition of \(T\) into the single \(s\) and a collection of canonical chains for all \(Q\)-vertices of \(T\). Assume that \(s = s_0 \neq r(T)\), and let \(P = s_0 q_0 s_1 q_1 \ldots s_{m-1} q_m s_m\) be the unique chain in \(T\) connecting \(s\) with the root of the tree \(r(T) = s_m\). Let \(s'_i\) denote a son of the vertex \(q_i\) in \(T\) distinguished from \(s_i\), \(i = 0, 1, \ldots, m - 1\). Removing from \(T\) the edges \(s_0 q_0, s_1 q_1, \ldots, s_{m-1} q_m\) and all edges of the form \(qp(q)\) for \(q \in Q \setminus \{q_0, q_1, \ldots, q_{m-1}\}\), we obtain the required partition of \(T\) into the single \(s = s_0\) and the chains \(s'_0 q_0 s_1, s'_1 q_1 s_2, \ldots, s'_{m-1} q_{m-1} s_m\), as well as all canonical chains for the vertices from \(Q \setminus \{q_0, q_1, \ldots, q_{m-1}\}\). It remains to note that the partition of the tree \(T\) is done in time \(O(k)\) because of the inequality \(m \leq |Q(T)| = k - 1\). \(\square\)