On Algebras of the Variety $\mathcal{B}_{1,1}$

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Abstract—By $\mathcal{B}_{1,1}$ we denote the variety of unary algebras of signature $f, g$ which is defined by the identity $fg(x) = x$. In this note it is proved that $\mathcal{B}_{1,1}$ is a cover for the variety $\mathcal{A}_{1,1}$, where $\mathcal{A}_{1,1}$ is the variety defined by the identities $fg(x) = x = gf(x)$. It is also shown that each endomorphism of a strongly connected algebra from $\mathcal{B}_{1,1}$ is an automorphism.

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1. INTRODUCTION

By $\mathcal{B}_{1,1}$ denote the variety of algebras with two unary operations $f$ and $g$ which is defined by the identity

$$fg(x) = x. \quad (1)$$

Algebras of this variety were investigated by some mathematicians from different points of view (see [1–5]).

Recall [2] that the bicyclic semigroup is the semigroup with two generators $f$ and $g$ and one defining relation $fg = e$, where $e$ is the identity element of this semigroup.

Let us remark that any act over the bicyclic semigroup can be treated as an algebra from $\mathcal{B}_{1,1}$. Obviously, $\mathcal{B}_{1,1}$ includes the variety $\mathcal{A}_{1,1}$ of algebras $\langle A, f, g \rangle$ satisfying the identities $fg(x) = x = gf(x)$. Up to now, algebras from this variety have been investigated quite well (see [1–4]).

In this paper the following theorem is proved.

**Theorem 1.** The variety $\mathcal{B}_{1,1}$ is a cover for $\mathcal{A}_{1,1}$ in the lattice of varieties of algebras with two unary operations.

It is well known that $EndA = AutA$ for any strongly connected commutative unary algebra $A$, where by $EndA$ and $AutA$ one means its endomorphism semigroup and automorphism group, respectively (see [6]).

This result can be generalised to the case of strongly connected algebras from the variety $\mathcal{B}_{1,1}$.

**Theorem 2.** Let $A$ be a strongly connected unary algebra and $A \in \mathcal{B}_{1,1}$. Then $EndA = AutA$.

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2. PRELIMINARY FACTS

In what follows, everywhere \( \mathbb{N} \) denotes the set of positive integers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

Let \( \langle A, \Omega \rangle \) be a unary algebra, \( f \in \Omega \) and \( a \in A \). By definition, we set \( f^0(a) = a, f^n(a) = f(f^{n-1}(a)) \) for each \( n \in \mathbb{N} \). By \( (a) \) we will denote the subalgebra generated by the element \( a \).

Suppose \( \{f, g\} \) is a signature consisting of two function unary symbols \( f, g \) and \( T(x) \) is a term which is built from \( f, g \), and a variable \( x \); then by \( T(x) \) we denote the term obtained from \( T(x) \) by omission all occurrences of the substring \( fg \). By \( (1) \), we have \( T(a) = T'(a) \) for any element \( a \) of an algebra \( A \) from the variety \( \mathcal{B}_{1,1} \).

It is clear that using this transformation, we can obtain a term whose notation does not contain a substring \( fg \). A term of this form we will call a reduced term.

The following lemma is an immediate consequence of the definition.

**Lemma 1.** An arbitrary reduced term \( T(x) \) can be represented in the form \( g^k f^s(x) \) \( (k, s \in \mathbb{N}_0) \).

Using Lemma 1, we easily see that Lemma 2 and Lemma 3 hold.

**Lemma 2.** Each element of a monogenic algebra \( (a) \) from the variety \( \mathcal{B}_{1,1} \) can be represented in the form \( g^k f^s(a) \) \( (k, s \in \mathbb{N}_0) \).

**Lemma 3.** Each algebra from the variety \( \mathcal{B}_{1,1} \) satisfies the quasi-identities

\[
Q_{i,k} : (\forall x, y)(g^k(y) = x \rightarrow (g^i f^i)(x) = x)(i, k \in \mathbb{N}, i \leq k).
\]

Suppose \( \langle A, \Omega, \rangle \) is a unary algebra, then we define the binary relation \( \sim \) on the set \( A \) by the rule \( a \sim b \iff (a) = (b) \) for any \( a, b \in A \). It is clear that \( \sim \) is an equivalence relation.

From the definition we can deduce the following assertion.

**Lemma 4.** If \( A \in \mathcal{B}_{1,1} \), then \( a \sim g^k(a)(k \in \mathbb{N}_0) \) for each element \( a \in A \).

3. MAIN RESULTS

**Proof of Theorem 1.** Assume that \( \mathcal{B}_{1,1} \) is not a cover of \( \mathcal{A}_{1,1} \). Then there is an algebra \( \langle A, f, g \rangle \) from the variety \( \mathcal{B}_{1,1} \) and an identity \( \sigma \) such that \( \langle A, f, g \rangle \) does not belong to \( \mathcal{A}_{1,1} \), \( \sigma \) is not a logical consequence of identity \( (1) \), and \( \langle A, f, g \rangle \) satisfies \( \sigma \).

By Lemma 1, without loss of generality it can be assumed that the identity \( \sigma \) is an equality of two reduced terms, i.e., \( \sigma \) is equivalent to an identity of the form

\[ a) (\forall x)g^r f^l(x) = g^k f^s(x) \quad (r, l, k, s \in \mathbb{N}_0) ; \]
\[ b) (\forall xy)g^r f^l(x) = g^k f^s(y) \quad (r, l, k, s \in \mathbb{N}_0). \]

Note that in either case at least one of the integers \( r, l, k, s \) is not equal to 0.

Now let us remark that since \( \langle A, f, g \rangle \in \mathcal{B}_{1,1} \), it follows that \( g : A \rightarrow A \) is an injective mapping and \( f : A \rightarrow A \) is a surjective mapping. Inasmuch as \( \langle A, f, g \rangle \notin \mathcal{A}_{1,1} \), we see that to obtain a contradiction it suffices to show that at least one of the mappings \( f \) or \( g \) is bijective.

We distinguish two cases.

**Case 1.** \( \sigma \) is equivalent to an identity of the form \( a ) \), i.e.,

\[ (\forall x)g^r f^l(x) = g^k f^s(x), \quad (r, l, k, s \in \mathbb{N}_0). \]

To be definite, assume that \( r \leq k \). Then \( \sigma \) is equivalent to the identity

\[ (\forall x)f^l(x) = g^{k-r} f^s(x), \]

Let \( k - r > 0 \). Then substituting \( g^l(x) \) for \( x \) in this identity, we see that the identity \( (\forall x)x = g^{k-r} f^s g^l(x) \) is also true in the algebra \( \langle A, f, g \rangle \). Therefore, the mapping \( g : A \rightarrow A \) is bijective.

If \( r = k \), then \( \sigma \) is equivalent to the identity \( (\forall x)f^l(x) = f^s(x) \).

Since \( \sigma \) is not a consequence of \( (1) \), we have \( l \neq s \). Assume that \( l < s \). Then substituting \( g^l(x) \) for \( x \), we see that \( (\forall x)x = f^{s-l}(x) \) holds in \( \langle A, f, g \rangle \), whence \( f \) is a bijection.