Comparative Analysis of Two Numerical Methods to Measure Hausdorff Dimension of the Fractional Brownian Motion

S. M. Prigarin¹, ²*, K. Hahn³**, and G. Winkler³***

¹Institute of Computational Mathematics and Mathematical Geophysics, Siberian Branch, Russian Academy of Sciences, pr. Akad. Lavrent’eva 6, Novosibirsk, 630090 Russia
²Novosibirsk State University, ul. Pirogova 2, Novosibirsk, 630090 Russia
³Institute of Biomathematics and Biometry, Helmholtz Zentrum Muenchen, Ingolstädter Landstraße 1, Neuherberg, 85764 Germany

Received February 8, 2007

Abstract—Using Monte Carlo simulation techniques, we look at statistical properties of two numerical methods (the extended counting method and the variance counting method) developed to estimate the Hausdorff dimension of a time series and applied to the fractional Brownian motion.

DOI: 10.1134/S1995423908020079

Key words: fractal set, Hausdorff dimension, extended counting method, variance counting method, generalized Wiener process, fractional Brownian motion.

1. INTRODUCTION

Fractals and multifractals are used as mathematical models of different physical objects and social phenomena in various fields of knowledge: biology, geophysics, financial mathematics, computer science, and others. Among the many research directions where the theory of fractals occupies the center stage are plasma physics, turbulence, porous media, biological systems, fractal antennas, image compression, computer graphics, data traffic, financial markets, functional magnetic resonance imaging, etc. Theoretical concepts such as deterministic and random fractals, multifractal singular measures, fractal dimension and fractional analysis play an important role in the present-day development of mathematics and physics (see, for example, [2–5, 7–9, 15]).

A significant research problem is a numerical analysis of fractal and multifractal structures, and in particular, the estimation of the fractal dimension of time series. In spite of the variety of developed numerical methods, the problem of their validity remains topical. In this paper, we apply the Monte Carlo method to study properties of two numerical methods designed to measure the Hausdorff dimension of a time series. The first method is a modification of the extended counting method originally destined for arbitrary sets in multidimensional spaces and adapted here for arbitrary time series (see Sec. 2). The second method is the variance counting method which is appropriate for random processes with stationary increments (see Sec. 3). As testing time series we use realizations of fractional Brownian motion with Hausdorff dimension in the interval [1, 2). Numerical modeling of the fractional Brownian motion is a nontrivial problem, which we shall not discuss here. An essential point is that in Monte Carlo simulation we use a numerical method that is exact in a probabilistic sense (see Sec. 4). Results of the Monte Carlo simulation and an analysis of the results are presented in Sec. 5.

*E-mail: sergeim.prigarin@gmail.com
**E-mail: hahn@helmholtz-muenchen.de
***E-mail: gwinkler@helmholtz-muenchen.de
2. THE EXTENDED COUNTING METHOD

An extended counting method [13, 14] was proposed as an alternative to the box counting method, to compute the Hausdorff dimension of a compact set in a finite-dimensional space. According to the extended counting method, the fractal dimension of a set $A \subset \mathbb{R}^2$ is estimated in the following way.

1. Fix a point $(x_0, y_0) \in \mathbb{R}^2$ and represent the space $\mathbb{R}^2$ as a sum of disjoint “atomic” squares $S_{ij}^{(1)}$ of a small size $d_1$, that is,

$$R^2 = \sum_{i,j=-\infty}^{+\infty} S_{ij}^{(1)}, \quad S_{ij}^{(1)} = [x_0 + id_1, x_0 + (i + 1)d_1) \times [y_0 + jd_1, y_0 + (j + 1)d_1).$$

We mark the atomic squares which contain points of the set $A$.

2. Fix a natural number $M$ and consider a set of (not necessarily disjoint) “exploratory” squares

$$S_{kn}^{(2)} = [x_0 + kd_1, x_0 + kd_1 + d_2) \times [y_0 + nd_1, y_0 + nd_1 + d_2),$$

where $d_2 = Md_1$ and $k$ and $n$ are integers. Every such exploratory square $S_{kn}^{(2)}$ consists of $M^2$ atomic squares $S_{ij}^{(1)}$ of which some may contain points of the set $A$. For a fixed exploratory square $S_{kn}^{(2)}$, we denote by $N_{kn}$ the number of atomic squares $S_{ij}^{(1)} \subset S_{kn}^{(2)}$ which contain points from $A$.

3. Find $N_{\text{max}} = \max_{kn} N_{kn}$ and compute the value

$$[d_1, d_2](A) = \frac{\log N_{\text{max}}}{\log d_2 - \log d_1} = \frac{\log N_{\text{max}}}{\log M}. \quad (1)$$

For suitable parameters $d_1$ and $d_2$, this number $\text{xdim}[d_1, d_2](A)$ will be interpreted as an estimator of the fractal dimension of the set $A$ (for details, see [13]). To apply the extended counting method, it is necessary to fix the following parameters: the origin of a fine grid $(x_0, y_0)$, the size of atomic squares $d_1$, and the coefficient $M$ that defines the size of exploratory squares, $d_2 = Md_1$. Below, we call $M$ an exploratory factor.

The main point of the extended counting method can be formulated as follows. The box counting method is applied to many subsets of a fractal set and the maximum of the subsets’ dimensions is taken as the fractal dimension of the set. On the other hand, the box counting method, as it is used for subsets, is extremely simplified (a box counting regression line is built only on the basis of 2 points).

2.1. Realization of the Extended Counting Method for a Time Series

To apply the extended counting method to a (finite) time series $x(ih)$, $i = 1, \ldots, N$, defined on a grid with step $h$, it is necessary to (i) define parameters of the method, and (ii) mark atomic squares which are assumed to be intersected by the graph of the time series. Solution of both of these problems is by no means evident. There is no unique procedure to reproduce a graph of a function with values defined on a fixed grid only. In our algorithms, we simply use a piecewise linear interpolation (another possible approach, for example, is to apply self-similar constructions). Moreover, to mark atomic squares intersected by the graph of the time series, it is reasonable to perform a preliminary scaling. This means that the extended counting method should be applied, not to the time series $x(ih)$ directly, but to a time series of the form

$$y(i) = Cx(ih).$$

Therefore, we have to introduce an additional parameter $C$ depending on $N$ and on the range $x(ih)$, $i = 1, \ldots, N$. To choose $C$, we use the following rule:

$$\frac{C[\max_i x(ih) - \min_j x(jh)]}{N - 1} = C_y.$$

NUMERICAL ANALYSIS AND APPLICATIONS Vol. 1 No. 2 2008