Dynamics of a Ring Micromechanical Gyroscope in the Forced-Oscillation Mode
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Received June 17, 2009

Abstract—The nonlinear effects of a vibrating micromechanical gyroscope with a ring resonator supported by a flexible torsion system are considered. A mathematical model of the thin elastic resonator forced oscillations is derived to account for the nonlinear properties of the resonator material. The resonator dynamics in slow variables measured by the device electronic circuit is investigated according to the Krylov–Bogolyubov averaging method. It is shown that the nonlinear elastic properties of the resonator material led to additional errors of the gyroscope, unstable branches of resonance curves, and quenching.

DOI: 10.1134/S2075108710010074

INTRODUCTION

The effect of base rotation on changes in the range of oscillation frequencies of thin elastic shells and rings was known by the end of the 19th century [1]. The physical phenomenon of the inertia inherent in elastic waves of free oscillations of an axisymmetric body, first explained in [2], has found practical application in the development of new types of gyroscopes [3–6]. The fundamentals of the theory of wave gyroscopes were laid in work [3], and a study of the error of such gyroscopes with various forms of vibrating resonators was done in [2–5]. It was shown that errors in manufacturing the resonator (e.g., variable density, thickness, and anisotropy in the elastic properties of the material), along with the nonlinear oscillations of the resonator studied in [3, 7, 8], lead to a split in the eigenfrequency of bending oscillations; this affects the wave pattern of the resonator oscillations and characterizes the accuracy of the gyroscope.

FORMULATION OF THE PROBLEM

Mathematical models of wave solid-state gyroscopes use different equations of the theory of distributed elastic shell (ring) systems [3, 7, 8]. The linear theory of shells based on the assumption of infinite smallness of the displacements of points of the body and a linear relationship between stress and strain of the body. As was shown in [9], two different processes of linearization, geometric and physical, lie at the heart of Hooke’s law, which expresses in general terms the connection between stress and strain for every point of the body.

Using linear theory, the main characteristics of the oscillating system (e.g., the frequency of bending oscillations and the gyro scale factor) are identified quite accurately. In the context of the linear theory, however, the phenomena observed during the experiment (the quenching, the dependence of oscillation frequencies on amplitude, and the drift of the gyroscope caused by small splittings of the oscillation frequency) cannot be explained.

These phenomena are inherent in nonlinear systems and can be explained by additional terms in a model of movement that involve finite (nonlinear) deformations of geometric nature, or a nonlinear relationship between stresses and deformation of the body that depends on the physical properties of the construction material.

In this article, the physical nonlinearity of the resonator material’s elastic properties is taken into account when deriving equations for the gyroscope’s motion. Assuming the oscillations of the resonator to be small, linear expressions for circular deformation are used. The impact of the geometric nonlinearity of the ring resonator’s oscillations was investigated in [3, 7].

Consider a ring vibrating micro gyroscope [6] whose resonator is a thin elastic ring (2), connected to base (1) with torsion bars (3) (Fig. 1). The thickness of the resonator is (h), while the axial line in the undeformed state is a circle of radius R. Resonator (1) is produced by lithography with torsion bars (2) of the elastic suspension and an electronic control contour. Oscillations of the resonator are excited and registered by control (3) and measuring electrodes (4).

The second basic form of the elastic oscillations of a thin elastic ring resonator is a superposition of two normal forms (the standing waves of oscillations), rotated relative to one another at angle $\pi/4$ (Fig. 2).
In the forced oscillation mode, the primary normal form of resonator oscillations is excited by means of electrostatic electrodes. The secondary normal form, whose antinodes coincide with the nodes of the primary normal form, appears when the base of the gyroscope is rotated. As was shown in [3, 6], the amplitude of the secondary normal form is proportional to the angular velocity of the base of the gyroscope in the open mode of operation, i.e., the gyroscope is a sensor of angular velocity.

Equations of motion. Let \( \mathcal{O}_{x'y'z'} \) be a right-hand Cartesian coordinate system associated with the base of the gyroscope and the plane \( \mathcal{O}_{x'y'} \), which contains the elastic ring, while \( r \) and \( \varphi \) are the polar coordinates in the plane \( \mathcal{O}_{x'y'} \). We denote by \( v \) and \( w \) the elastic displacements of the ring resonator element in the circular and the radial directions, respectively. Assume that the axis of the resonator mounting revolves around axis \( oz \) with slowly changing angular velocity \( \Omega \), which will henceforth be considered small compared to characteristic oscillation frequency of the resonator \( \omega \). In this case, the specific kinetic energy of the thin elastic ring resonator per unit of length of the axial line is expressed as

\[
T = \rho RS[(v + \Omega(w + R))^2 + (\Omega v - w)^2]/2, \tag{1}
\]

where \( \rho \), \( R \), and \( S \) are the material density, radius, and cross-sectional area of the resonator, respectively. Here and below, the derivative of time \( t \) is denoted by a dot.

Assume that the material properties of the resonator obey the nonlinear Hooke’s law [9]:

\[
\sigma = E(e - a_3 \varepsilon^3), \tag{2}
\]

where \( \sigma \) are the stresses in the resonator upon circular deformation \( \varepsilon \); \( E \) is Young’s modulus, and \( a_3 \) is a dimensionless elastic modulus.

The deformation energy of the resonator’s unit volume with (2) taken into account has the form [10]:

\[
\Pi = \int_0^e \sigma de = \frac{1}{2} E \left[ e^2 - a_3 \frac{e^4}{2} \right] . \tag{3}
\]

The expression for circular deformation is given in [10]:

\[
e = \frac{1}{R} (v' + w) - \zeta \frac{v' - w''}{R}. \tag{4}
\]

Here, the prime denotes the derivative with respect to circular coordinate \( \varphi \), and \( \zeta \) is a coordinate measured from the midline of the resonator in the direction of the outward normal \((-h/2 \leq \zeta \leq h/2)\).

After substituting (4) in (3) and integrating over the square of the resonator cross-section, we obtain the specific potential energy:

\[
P = \frac{1}{2} \int \frac{E S}{R} (v' + w)^2 + \frac{EI}{R^2} (v' - w'')^2 \left[ \frac{S}{2R} (v' + w)^4 + \frac{3Ih^2}{40R^5} (v' - w'')^4 \right]. \tag{5}
\]

Here, \( I \) is a moment of inertia of the ring cross-section, \( c \) is a coefficient characterizing the rigidity of the supporting torsion bars, and \( q(t, \varphi) \) is the specific electric power of the resonator’s oscillation excitation:

\[
q(t, \varphi) = \frac{\varepsilon_0 d (\frac{U}{d})^2}{2}, \tag{6}
\]

where \( \varepsilon_0 = 8.854 \times 10^{-12} \text{C/(N m}^2) \) is an electric constant; \( l \) is the height of the electrode; \( d = d_0 + w(t, \varphi) \) is the gap between the ring resonator and the electrodes; \( d_0 \) is the gap between the non-deformable resonator and the electrodes; and \( U = U(t, \varphi) \) is the potential difference between the power electrodes and the resonator, which can be written as follows: