Some geometric aspects of $\mathbb{C}P^{N-1}$ maps

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Abstract. In this talk we introduce a Weierstrass-like system of equations corresponding to $\mathbb{C}P^{N-1}$ fields in two dimensions. Then using this representation we introduce a vector $\mathbf{r}$ in $R^{N^2-1}$ and treating this vector as the radius vector of a surface immersed in $R^{N^2-1}$ we discuss to what extent the associated metric describes the geometry of the $\mathbb{C}P^{N-1}$ maps. We show that for the holomorphic maps – the correspondence is exact; while for the more general fields we have to go beyond the Weierstrass system and add extra terms.

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1 Introduction

In this talk I want to discuss two interesting ideas, and then relate them to each other. Moreover, I will then show how these ideas can be further generalised.

The work described in this talk has been done in collaboration with M. Grundland. The details of the work will be given elsewhere [1].

1.1 Harmonic maps [2]

The first topic involves $\mathbb{C}P^1$ harmonic maps. These are maps

$$S^2 \rightarrow \mathbb{C}P^1 \sim S^2$$

given by the stationary points of the energy

$$\mathcal{L} = \int L \, dx \, dy,$$

where

$$L = \frac{1}{4} (D_\mu z)^\dagger \cdot D_\mu z,$$

and where, in the general case of $\mathbb{C}P^{N-1}$, $z$ is a vector field of $N$ components, $z = (z^1, ..., z^N)$, which satisfies

$$z^\dagger \cdot z = 1.$$  (4)

Here $\mu = 1, 2$, of course, and denotes the space coordinates $x$ and $y$. In the $\mathbb{C}P^1$ case we can introduce a complex field $W$

$$z = \frac{(1, W)}{\sqrt{1 + |W|^2}}.$$  (6)

Then, the Euler Lagrange equations describing harmonic maps are given by

$$\partial \bar{\partial} W - 2W \frac{\partial W \bar{\partial} W}{|W|^2 + 1} = 0$$

where $W = W(\zeta, \bar{\zeta})$ and

$$\partial = \frac{\partial}{\partial (x + iy)} = \frac{\partial}{\partial \zeta}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{\zeta}}.$$  (8)

1.2 Weierstrass system [3]

Here we consider 2 complex functions $\psi = \psi(\zeta, \bar{\zeta})$ and $\phi = \phi(\zeta, \bar{\zeta})$, which satisfy

$$\partial \psi = p \phi, \quad \bar{\partial} \phi = -p \psi, \quad p = |\phi|^2 + |\psi|^2.$$  (9)

Note that we have not specified $\partial \psi$, nor $\bar{\partial} \phi$.

A natural question then arises. Are these two problems related? Obviously the answer is YES. To see this put

$$W = \frac{\psi}{\phi}$$  (10)
and
\[ \psi = W \frac{\partial W}{1 + |W|^2}, \quad \phi = \frac{\partial W}{1 + |W|^2}. \] (11)
and so find that (7) and (9) are equivalent.

Moreover, we can introduce 3 real quantities
\[ X_1 = i \int_\gamma [\bar{\psi} + \phi] d\zeta - [\psi + \phi^2] d\bar{\zeta}, \]
\[ X_2 = \int_\gamma [\bar{\psi} - \phi^2] d\zeta + [\psi - \phi^2] d\bar{\zeta}, \]
\[ X_3 = -2 \int_\gamma \bar{\psi} d\zeta + \bar{\psi} d\bar{\zeta}, \] (12)
where \( \gamma \) is any curve from a fixed point to \( \zeta \).

Then, it is easy to show that if \( \psi \) and \( \phi \) satisfy (9) then \( X_i \) do not depend on the details of the curve \( \gamma \) but only on its endpoints.

Furthermore, if we treat \( X_i \) as components of a vector \( \mathbf{r} = (X_1, X_2, X_3) \) and introduce the metric
\[ g_{\zeta \bar{\zeta}} = (\partial \mathbf{r}, \partial \mathbf{r}), \quad g_{\zeta \zeta} = (\partial \mathbf{r}, \partial \mathbf{r}), \quad g_{\zeta \bar{\zeta}} = (\partial \mathbf{r}, \partial \mathbf{r}) \] (13)
we find that, for fields which solve (7) on \( S^2 \), only \( g_{\zeta \bar{\zeta}} \) is non-zero and is given by
\[ g_{\zeta \bar{\zeta}} = \frac{|\partial W|^2}{(1 + |W|^2)^2} = |Dz|^2, \] (14)
where \( D \) denotes the \( D_\mu \) derivative (5) but evaluated with respect to \( \zeta \). Note that (14) is a term in the general expression for the energy of the \( CP^1 \) map. However, as all harmonic maps on \( S^2 \) satisfy \( W = W(\zeta) \) [2], \( g_{\zeta \bar{\zeta}} \) is the energy. \(^1\)

Can we generalise this to \( CP^{N-1} \)?

2 \( CP^{N-1} \) case

2.1 General considerations [2]

In the \( CP^{N-1} \) case we can put
\[ z_i = \frac{f_i}{|f|}, \quad i = 1, \ldots, N, \] (15)
and introduce \( W_k = \frac{f_k}{|f|} \). The energy is still given by (3) and the Euler Lagrange equations become
\[ \left( 1 - \frac{f f^\dagger}{|f|^2} \right) \left[ \partial \bar{\partial} f - \partial f \frac{f f^\dagger}{|f|^2} - \bar{\partial} f \frac{f f^\dagger}{|f|^2} \right] = 0. \] (16)

\(^1\) We are assuming here that we are not dealing with anti-holomorphic maps, as then \( g_{\zeta \bar{\zeta}} = 0 \); in this case we exchange the roles of \( \zeta \) and \( \bar{\zeta} \).

Holomorphic solutions imply \( \partial f = 0 \) but, for \( N > 2 \) there exist nonholomorphic solutions of (16).

To construct the generalised Weierstrass system we note that (16) can be rewritten as
\[ [\partial \bar{\partial} P, P] = 0 \quad \text{where} \quad P = \frac{1}{A} f f^\dagger, \] (17)
where \( A = f f^\dagger \). Thus we have
\[ \partial K + \bar{\partial} M = 0, \quad \text{with} \quad K = [\partial P, P], \quad M = [\bar{\partial} P, P] \] (18)
The explicit form of matrices \( K \) and \( M \) is given by\(^2\)
\[ K_{ij} = \frac{1}{A^2} \left[ \bar{f}_k f_k \bar{\partial} f_i \bar{\partial} f_j - \bar{f}_k f_k \bar{f}_i \bar{f}_j, \right. \]
\[ M_{ij} = \frac{1}{A^2} \left[ f_k f_k \partial f_i \partial f_j - f_k f_k \bar{f}_i \bar{f}_j, \right. \]
\[ = f_k f_k \bar{f}_i \bar{f}_j - f_k f_k f_i \bar{f}_j, \] (19)

which we next rewrite as
\[ K_{ij} = \bar{f}_j \bar{\phi}_{\bar{z}}^2 - f_i \varphi_j^2, \quad M_{ij} = \bar{f}_j \varphi_i^2 - f_i \bar{\phi}_{\bar{z}}^2, \] (20)
where we have defined
\[ \varphi_i^2 = \frac{1}{A^2} \bar{f}_k f_{ki}, \quad \Phi_i^2 = \frac{1}{A^2} f_k \bar{f}_{ki}, \] (21)

and
\[ F_{ij} = f_i \bar{\partial} f_j - f_j \bar{\partial} f_i, \quad \bar{F}_{ij} = f_i \bar{\partial} f_j - f_j \bar{\partial} f_i, \] (22)

Note that we have two constraints
\[ \bar{f}_k \varphi_k^2 = 0, \quad f_k \Phi_k^2 = 0, \] (23)
which allows us to consider as independent only \( \varphi_2, \ldots, \varphi_N \).

At the same time, as we have said before, we can set, say, \( f_1 = 1 \) and so we end up with
\[ \varphi_i^2 = \frac{1}{A^2} \left[ (1 + f_k f_k) \bar{\partial} f_i - f_k (f_k \bar{\partial} f_i) \right], \] (24)

where all the sums over repeated indices run over \( k = 2, \ldots, N \).

Note that the \( CP^{N-1} \) system has more conserved quantities. In fact, we can drop the \( \Phi \) terms in the expressions for \( K \) and \( M \) in (21) and we still have our conservation laws; namely, we can define
\[ K'_{ij} = -f_i \varphi_j^2, \quad M'_{ij} = \varphi_i^2 \bar{f}_j \] (25)
and then note that we still have
\[ \partial K' + \bar{\partial} M' = 0. \] (26)
This is easy to check the validity of (27) by using (16).\(^2\) We assume summation convention.