Real Hamiltonian forms of Hamiltonian systems

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Abstract. We introduce the notion of a real form of a Hamiltonian dynamical system in analogy with the notion of real forms for simple Lie algebras. This is done by restricting the complexified initial dynamical system to the fixed point set of a given involution. The resulting subspace is isomorphic (but not symplectomorphic) to the initial phase space. Thus to each real Hamiltonian system we are able to associate another nonequivalent (real) one. A crucial role in this construction is played by the assumed analyticity and the invariance of the Hamiltonian under the involution. We show that if the initial system is Liouville integrable, then its complexification and its real forms will be integrable again and this provides a method of finding new integrable systems starting from known ones. We demonstrate our construction by finding real forms of dynamics for the Toda chain and a family of Calogero–Moser models. For these models we also show that the involution of the complexified phase space induces a Cartan-like involution of their Lax representations.

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1 Introduction

Recently the so-called complex Toda chain (CTC) was shown to describe $N$-soliton interactions in the adiabatic approximation [1–3]. The complete integrability of the CTC is a direct consequence of the integrability of the real (standard) Toda chain (TC); it was also shown that CTC allows several dynamical regimes that are qualitatively different from the one of real Toda chain [2]. These results as well as the hope to understand the algebraic structures lying behind the integrability of CTC (such as, e.g. Lax representation) were the stimulation for the present work.

We start from a standard (real) Hamiltonian system $\mathcal{H} \equiv \{\omega, H, \mathcal{M}\}$ with $n$ degrees of freedom and Hamiltonian $H$ depending analytically on the dynamical variables. It is known that such systems can be complexified and then written as a Hamiltonian system with $2n$ (real) degrees of freedom. Our main aim is to show that to each compatible involutive automorphism $\tilde{C}$ of the complexified phase space we can relate a real Hamiltonian form of the initial system with $n$ degrees of freedom. Just like to each complex Lie algebra one associates several inequivalent real forms, so to each $\mathcal{H}$ we associate several inequivalent real forms $\mathcal{H}_R \equiv \{\omega_R, H_R, \mathcal{M}_R\}$. Like the initial system $\mathcal{H}$, the real form is defined on a manifold $\mathcal{M}_R$ with $n$ real degrees of freedom. Provided $\tilde{C}(H) = H$ the dynamics on the real form will be well defined and will coincide with the dynamics on $\mathcal{M}_C$ restricted to $\mathcal{M}_R$. We show that if the initial system $\mathcal{H}$ is integrable then its real Hamiltonian forms will also be integrable. We pay special attention to the connection with integrable systems and the possibility they offer to define a class of new integrable systems starting from an initial one. Recently a procedure to obtain new integrable systems by composing known integrable ones has been elaborated in the framework of coproducts [4]. Here we are not concerned with this approach.

Examples of indefinite-metric Toda chain (IMTC) have already been studied by Kodama and Ye [5]. In particular they note that while the solutions of the TC model are regular for all $t$, the solutions of the IMTC model develop singularities for finite values of $t$. Particular examples of non-standard (or “twisted”) real forms of 1 + 1-dimensional Toda field theories have already been studied by Evans and Madsen [6] in connection with the problem of positive kinetic energy terms in the Lagrangian description and with emphasis on conformal WZNW models.

The approach we follow here is different and more general than the ones in [5,6]. Its main ideas were reported in [7]; here we elaborate the proofs the details and provide new classes of examples.

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2 Complexified Hamiltonian dynamics

We start with a real Hamiltonian system with \( n \) degrees of freedom \( \mathcal{H} \equiv \{ \mathcal{M}^{(n)}, H, \omega \} \) where \( \mathcal{M}^{(n)} \) is a 2\( n \) dimensional vector space and

\[
\omega = \sum_{k=1}^{n} dp_k \wedge dq_k. \tag{1}
\]

Let’s consider its complexification:

\[
\mathcal{H}^C \equiv \{ \mathcal{M}^{(n)}, H^C, \omega^C \} \tag{2}
\]

where \( \mathcal{M}^{(2n)} \) can be viewed as a linear space \( \mathcal{M} \) over the field of complex numbers:

\[
\mathcal{M}^{(2n)} = \mathcal{M}^{(n)} \otimes i \mathcal{M}^{(n)}. \]

In other words the dynamical variables \( p_k, q_k \) in \( \mathcal{M}^{(n)} \) now may take complex values. We assume that observables \( F, G \) and the Hamiltonian \( H \) are real analytic functions on \( \mathcal{M} \) and can naturally be extended to \( \mathcal{M}^{(2n)} \).

The complexification of the dynamical variables \( F, G \) and \( H \) means that they become analytic functions of the complex arguments:

\[
p_k^C = p_k, 0 + ip_k, 1, \quad q_k^C = q_k, 0 + iq_k, 1, \quad k = 1, \ldots, n \tag{3}
\]

and we can write:

\[
H^C = H(p_k^C, q_k^C) = H_0 + iH_1. \tag{4}
\]

The same goes true also for the complexified 2–form:

\[
\omega^C = \sum_{k=1}^{n} dp_k^C \wedge dq_k^C = \omega_0 + i\omega_1. \tag{5}
\]

Note that each of the symplectic forms \( \omega_0 \) and \( \omega_1 \) are non-degenerate. However the linear combination \( \omega^C = \omega_0 + i\omega_1 \) can be written down in the form:

\[
\omega = \sum_{k=1}^{n} (dp_k, 0, dp_k, 1, dq_k, 0, dq_k, 1) B_0 \begin{pmatrix} dp_k, 0 \\ dp_k, 1 \\ dq_k, 0 \\ dq_k, 1 \end{pmatrix}
\]

where the matrix \( B_0 \)

\[
B_0 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & i & 0 & -1 \\ -1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \end{pmatrix}
\]

obviously has the property \( B_0^2 = 0 \).

**Remark 1.** The kernel of \( \omega^C \) is spanned by the antiholomorphic vector fields. We could also choose the antiholomorphic (anti-analytic) functions in the complexification procedure. This would lead to equivalent results.

Obviously, \( \dim \mathcal{M}^C = 4n \) and therefore \( \mathcal{H}^C \) may be considered as a real dynamical system with \( 2n \) degrees of freedom. To elaborate on this, we start from the complexified equations of motion:

\[
\frac{dp_k^C}{dt} = -\frac{\partial H^C}{\partial q_k^C}, \quad \frac{dq_k^C}{dt} = \frac{\partial H^C}{\partial p_k^C}. \tag{6}
\]

The right hand side of (6) contain the partial derivatives of both \( H_0 \) and \( H_1 \). Since we assumed analyticity, \( H_0 \) and \( H_1 \) will satisfy the Cauchy-Riemann equations:

\[
\frac{\partial H_0}{\partial q_k^C} = \frac{\partial H_1}{\partial p_k^C}, \quad \frac{\partial H_0}{\partial p_k^C} = -\frac{\partial H_1}{\partial q_k^C}. \tag{7}
\]

Analogous formulae hold for the derivatives with respect to \( p_k^C \) and \( q_k^C \). Thus all terms in the right hand sides of (6) are equal to:

\[
\frac{dp_k^C}{dt} = -\frac{\partial H_0}{\partial q_k^C}, \quad \frac{dq_k^C}{dt} = \frac{\partial H_0}{\partial p_k^C}. \tag{8}
\]

Obviously (9) are standard Hamiltonian equations of motion for a dynamical system with \( 2n \) degrees of freedom corresponding to:

\[
H_0 = \text{Re} \, H^C(p_k^C, q_k^C), \tag{10}
\]

\[
\omega_0 = \text{Re} \, \omega^C = \sum_{k=1}^{n} (dp_k, 0 \wedge dq_k, 0 - dp_k, 1 \wedge dq_k, 1). \]

We denote the related real dynamical vector field by \( \Gamma_0 \).

The system (6) allows a second Hamiltonian formulation with:

\[
H_1 = \text{Im} \, H^C(p_k^C, q_k^C), \tag{11}
\]

\[
\omega_1 = \text{Im} \, \omega^C = \sum_{k=1}^{n} (dp_k, 0 \wedge dq_k, 1 + dp_k, 1 \wedge dq_k, 0)
\]

and also real dynamical vector field \( \Gamma_1 \). Due to the analyticity of \( H^C \) Cauchy-Riemann equations yield that these two vector fields actually coincide:

\[
\Gamma_0 = \Gamma_1.
\]

So \( \Gamma_0 \) is a bi-Hamiltonian vector field and the corresponding recursion operator is:

\[
T = \omega_0^{-1} \circ \omega_1 \quad \tag{12}
\]

\[
= -\frac{\partial}{\partial p_1} \otimes dp_0 + \frac{\partial}{\partial p_0} \otimes dp_1 - \frac{\partial}{\partial q_1} \otimes dq_0 + \frac{\partial}{\partial q_0} \otimes dq_1
\]

\[
T^2 = -1.
\]