Test of universality in the Ising spin glass using high temperature graph expansion

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Abstract. We calculate high-temperature graph expansions for the Ising spin glass model with 4 symmetric random distribution functions for its nearest neighbor interaction constants \(J_{ij}\). Series for the Edwards-Anderson susceptibility \(\chi_{EA}\) are obtained to order 13 in the expansion variable \((J/(k_BT))^2\) for the general \(d\)-dimensional hyper-cubic lattice, where the parameter \(J\) determines the width of the distributions. We explain in detail how the expansions are calculated. The analysis, using the Dlog-Padé approximation and the techniques known as M1 and M2, leads to estimates for the critical threshold \((J/(k_BT))_{c}\) and for the critical exponent \(\gamma\) in dimensions 4, 5, 7 and 8 for all the distribution functions. In each dimension the values for \(\gamma\) agree, within their uncertainty margins, with a common value for the different distributions, thus confirming universality.

PACS. 05.70.Jk Critical point phenomena – 75.10.Nr Spin-glass and other random models

1 Introduction

In 1975, Edwards and Anderson (EA) [1] introduced a model for the theoretical study of spin glasses (SG) [2,3], which has started modern spin glass theory and has been of continued interest until today. Here we discuss the classical Ising case: The magnetic moments are represented by ‘spin’ variables \(\{s_i, i = 1, 2, \ldots, N\}\), which can assume the values \(s_i = \pm 1\) and are located on the sites \(\{i\}\) of the \(d\)-dimensional hyper-cubic lattice. During our calculations we use a finite number of lattice sites \(N\) but eventually we are interested in the thermodynamic limit, \(N \to \infty\). The spins’ interaction is governed by the Hamiltonian

\[
\mathcal{H}(\{J_{ij}\})(\{s_i\}) = - \sum_{\langle ij \rangle} J_{ij} s_i s_j - h_0 \sum_{i=1}^{N} s_i,
\]

where \(\sum_{\langle ij \rangle}\) denotes the sum over all pairs of nearest neighbor lattice sites \(\langle ij \rangle\), which we also call the lattice bonds, and the spin interaction constants \(J_{ij}\) are chosen at random from a symmetric probability distribution, which is the same for all bonds. The external magnetic field \(h_0\) is needed to define thermodynamic quantities as derivatives with respect to it, but apart from that, we concentrate on the case \(h_0 = 0\). The Hamiltonian’s index \(\{J_{ij}\}\) indicates that we deal with quenched disorder, i.e. the thermodynamic average for any observable is performed for a fixed set of coupling constants \(\{J_{ij}\}\). The configurational average of measurable thermodynamic quantities, over the random variables, is performed subsequently. For self averaging quantities this leads to expressions of what could be measured in experiments. We denote the thermodynamic average of any observable \(A(\{s_i\})\) by \(\langle A \rangle_{T}\), and the configurational average of any function \(X(\{J_{ij}\})\) by \([X]_{R}\).

The EA model neglects the details of the microscopic interaction between the spins, but exhibits the two essential ingredients that lead to the interesting features of spin glasses: Quenched disorder and frustration. Since little has been proved exactly for short ranged spin glass models, we assume what today is generally accepted, based on analytical and numerical evidence: Above the system’s lower critical dimension \(d_{c}\), whose value is controversial but agreed to be between 2 and 3 [4,5], it undergoes a continuous transition at a non-zero critical temperature \(T_{c}\) to a low temperature spin glass phase. This phase is characterized by broken spin-flip symmetry, i.e. a non-zero Edwards-Anderson order parameter

\[
q_{EA} = \frac{1}{N} \sum_{i=1}^{N} \left[\langle s_i^2 \rangle_{T} \right]_{R}.
\]
The upper critical dimension, above which mean field behavior becomes dominant, is believed to be \( d_u = 6 \) [6,7].

As the temperature \( T \) approaches \( T_c \) from above, we expect the susceptibility associated with \( q_{EA} \), the Edwards-Anderson susceptibility,

\[
\chi_{EA} = \frac{1}{N} \sum_{i,j=1}^{N} \left[ \langle s_i s_j \rangle \right]_{R},
\]

(3)
to exhibit a power law divergence, \( \chi_{EA} \sim (T_c - T)^{-\gamma} \), characterized by the critical exponent \( \gamma \). In the present study we use series expansions to investigate this behavior. Both \( q_{EA} \) and \( \chi_{EA} \) are related to configurational averages of higher order logarithmic derivatives of the partition function \( -\frac{\partial^m \ln Z}{\partial b_0^m} |_{b_0=0} \) with respect to the external magnetic field. Those relations become linear in the thermodynamic limit [8].

The renormalization group theory [9] in dimension \( d = 6 - \epsilon \) predicts the universality of \( \gamma \) and of other exponents, related to it by scaling relations. The universality classes should be set by the dimensionialities of space and of the spin variables, and not by details of the distribution functions.

Series expansion has been used in the past to study the spin glass transition [8,10–13] and the results support the statements mentioned above. Our renewed interest in the problem awoke with a series of studies [14–20] that found, based on computer simulations, that the critical exponents vary with the probability distribution for the quenched disorder in the coupling constants \( J_{ij} \). This is in clear violation of universality and not sufficiently explained by theory.

Undoubtedly, many of the enormous complications and features observed in the study of spin glasses arise from the disorder inherent in these systems. They gave the model the reputation of being one of the toughest subjects in computational physics. Simulations are here directly impacted by long relaxation times, memory effects, hysteresis, the rugged energy landscape with many metastable states and the huge parameter space over which to average.

The technique of series expansion comes with two immediate advantages: The averaging over the randomness can be done \emph{exactly}, and the series can, given the availability of graph data, be obtained in general dimension. The subsequent analysis is still done in each dimension separately, but results generally get more reliable with increasing dimension, while simulations become increasingly expensive in their computational demands. The previous series expansion studies of the Ising spin glass used only the bimodal random distribution of \( J_{ij} = \pm J \), limiting their use in the comparison with the claims of violated universality. In the present study we extend the research by addressing several other symmetric distribution functions, each with a variable width determined by the parameter \( J \). We use the same distributions as Bernardi and Campbell in [15], except for the exponential distribution, which is excluded for reasons given in Section 6.

Introducing additional notations and the random distribution functions in Section 2, we give a detailed explanation of the series generation in Sections 3 and 4, which should allow the interested reader to follow each step. As an example, we actually show the complete calculation of a fourth order series in Section 5. In Section 6 we present our general-dimension series in full, accompanied by some discussion of accuracy checks. Our series analysis and final results are described in Section 7 and we finish with our conclusions in Section 8.

2 Further notations and definitions

With \( \beta = \frac{1}{k_B T} \), where \( k_B \) denotes Boltzmann’s constant and \( T \) the absolute temperature, the ensemble average of an observable \( A \) is calculated by

\[
\langle A \rangle_T = \frac{\text{Tr} (A e^{-\beta H})}{Z} = \frac{\text{Tr} (A e^{-\beta H})}{\text{Tr} (e^{-\beta H})},
\]

(4)

where the partition function \( Z \) appears in the denominator. Here the trace (Tr) is a shorthand for summing over all possible values of the spins’ \( \{ s_i \} \) configuration

\[
\text{Tr} X = \text{Tr} X(\{ s_i \}) = \sum_{s_i=\pm 1} \cdots \sum_{s_N=\pm 1} X(\{ s_i \}).
\]

(5)

The free energy per site \( F \) is obtained from \( Z \) by

\[
F = \frac{1}{N} F_N \equiv -\frac{1}{\beta N} [\ln Z]_R.
\]

(6)

Since the interaction constants \( J_{ij} \) appear only in products with \( \beta \), it is convenient to use \( \kappa_{ij} = \beta J_{ij} \) as the argument of the distribution functions introduced below. If \( J^2 \) is some measure of \( |J_{ij}|^2 \), then we also use \( \kappa = \beta J \) as expansion variable, at least temporarily. Since only even powers of \( \kappa \) remain, we eventually use \( x = \kappa^2 \) as the expansion variable in our high temperature series. Likewise we use \( x_c = (J/\langle k_B T \rangle)^2 \) to denote the critical threshold.

In the general case of a continuous probability distribution \( P(z) \), the configurational average is the nested integral

\[
[X]_R = \int_{-\infty}^{-\infty} \cdots \int_{-\infty}^{-\infty} X(\{ z_{ij} \}) \prod_{\langle ij \rangle} (P(z_{ij}) \, dz_{ij}).
\]

(7)

For the bimodal random distribution the coupling constants \( \kappa_{ij} \) for nearest neighbor pairs randomly assume only values of either \( +\kappa \) or \( -\kappa \), so the latter integral can be written as the nested sum

\[
[X]_R = \frac{1}{2^{N^2}} \sum_{\{ \kappa_{ij} \}_{ij} = \pm \kappa} X(\{ \kappa_{ij} \}),
\]

(8)

where a normalization factor of \( 1/2^s \) stems from each \( \kappa_{ij} \) in the sum. In the \( d \)-dimensional hyper-cubic lattice with