Spontaneous breaking of chiral symmetry, and eventually of parity, in a $\sigma$-model with two Mexican hats

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Received: 11 August 2009 / Revised: 11 September 2009 / Published online: 3 December 2009
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Abstract A $\sigma$-model with two linked Mexican hats is discussed. This scenario could be realized in low-energy QCD when the ground state and the first excited (pseudo)scalar mesons are included, and where not only in the subspace of the ground states, but also in that of the first excited states, a Mexican hat potential is present. This possibility can change some basic features of a low-energy hadronic theory of QCD. It is also shown that spontaneous breaking of parity can occur in the vacuum for some parameter choice of the model.

1 Introduction

The ‘Mexican hat’ potential allows for a simple and intuitive description of the phenomenon of spontaneous symmetry breaking. For this reason it has been widely used—in a variety of versions—in both condensed matter and hadron physics; see for instance Ref. [1] and references therein.

In the context of Quantum Chromodynamics (QCD) nearly massless $N_f^2 - 1$ (3 pions in the case $N_f = 2$, where $N_f$ is the number of light-quark flavors) emerge as (quasi) pseudoscalar Goldstone bosons as a consequence of spontaneous breaking of chiral symmetry: $U_R(N_f) \times U_L(N_f) \to SU_V(N_f)$. In the context of a linear $\sigma$-model this spontaneous breaking is induced by a negative squared mass of the scalar and pseudoscalar mesons. This feature is responsible for the typical Mexican hat form of the mesonic potential.

In this work, beyond the ground-state (pseudo)scalar mesons, we also consider the first excited (pseudo)scalar states and we investigate the case in which also in this sector a negative squared mass is present. As we shall argue, for some parameter choice this possibility cannot be excluded and leads to a more complicated scenario, in which ground state and first excited scalar and pseudoscalar mesons mix.

Moreover, for some parameter choice it is possible that also one neutral pseudoscalar pionic field condenses, thus realizing a spontaneous symmetry breaking of parity.

The paper is organized as it follows: we first briefly review the properties of the Mexican hat potential and its emergence from an hadronic model of QCD. We then turn to the case of two linked Mexican hats and discuss the consequences of this assumptions. First, the parameter range in which only spontaneous breaking of chiral symmetry take place is studied. Then, the parameter range in which also spontaneous breaking of parity occurs is investigated. In the end, the conclusions are briefly outlined.

2 Mexican hat

In its simplest form the Mexican hat potential is written in terms of two real scalar fields $\sigma$ and $\pi$:

$$V_{\text{MH}} = \frac{\lambda}{4} (\sigma^2 + \pi^2 - F^2)^2 = \frac{\lambda}{4} (\phi^* \phi - F^2)^2,$$

where in the last step the complex scalar field $\phi = \sigma + i\pi$ has been introduced. The requirement $\lambda \geq 0$ ensures that the potential is bounded from below. Let us assume that—as in QCD, see below—$\sigma$ represents a scalar field ($\sigma \equiv \sigma(t, x) \to \sigma(t, -x)$ under parity transformation $P$) while $\pi$ represents a pseudoscalar field ($\pi \equiv \pi(t, x) \to -\pi(t, -x)$ under $P$).

Note that the quadratic (mass) term of the Mexican hat potential reads $-\frac{1}{2} F^2 \phi^* \phi$, i.e. it has a negative coefficient as long as $F$ is a real number, which corresponds to an imaginary mass for both the $\sigma$ and the $\pi$ fields. For this reason one can immediately deduce that the point $\phi = 0$ does not correspond to the minimum of the potential. Moreover, an expansion around this point is instable.

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The potential $V_{\text{MH}}$ is symmetric under $SO(2) \sim U(1)$ (denoted as chiral) transformation, namely:

$$
\begin{pmatrix}
\sigma \\
\pi
\end{pmatrix} \rightarrow
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\sigma \\
\pi
\end{pmatrix}
$$

or

$$
\varphi \rightarrow e^{-i\theta} \varphi.
$$

(2)

The model does not have a unique minimum: all the points $\varphi_{\text{min}} = F e^{i\theta}$ for each $\theta \in [0, 2\pi)$ are minima. If no other information is given, each one of these minima can be in principle realized. However, we assume that a small perturbation, which breaks chiral symmetry but does not break parity, $V_{\text{MH}} \rightarrow V_{\text{MH}} - \varepsilon \sigma$ with $\varepsilon \in 0^+$, takes place: as a consequence, the only realized minimum is $\varphi_{\text{min}} = F$. (A change of sign of $\varepsilon$ would simply provide the equivalent solution $-\varphi_{\text{min}}$.) When evaluating the fluctuations around the minimum $\varphi_{\text{min}} = F$, one obtains a scalar, massive $\sigma$ meson with $M_\sigma^2 = 2\lambda F^2$ and a pseudoscalar, massless Goldstone boson $\pi$. The chiral symmetry of the model is not realized as a degeneracy of the particle spectrum because the minimum (i.e. the vacuum) is not left invariant by this transformation: spontaneous breaking of chiral symmetry has taken place and the field $\pi$ is the corresponding Goldstone boson.

3 QCD origin of the Mexican hat

For the purpose of this paper we briefly recall how the Mexican hat potential describes the spontaneous breaking of chiral symmetry which is observed in the context of low-energy QCD. The matrix $\Phi = S + iP$ includes $N_f^2 - 1$ pseudoscalar fields, $S = S^a t^a$ and $P = P^a t^a$ where the matrices $t^a$ with $a = 1, \ldots, N_f^2 - 1$ are the generators of $SU(N_f)$ (with $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$) and $t^0 = \frac{1}{2N_f^2} \mathbf{1}_{N_f^2}$. Upon chiral transformation $U_R(N_f) \times U_L(N_f)$ the field $\Phi$ transforms as $\Phi \rightarrow L \Phi R^\dagger$ with $L \in U_L(N_f)$ and $R \in U_R(N_f)$. The transformation in flavor space $SU_V(N_f)$ is obtained by setting $L = R = U_V$, where $U_V$ is a $SU(N_f)$ matrix. The transformation $SU_A(N_f)$ is obtained by setting $L = R^\dagger = U_A$, where $U_A$ is a $SU(N_f)$ matrix. (Note, however, that this set of transformations does not form group for $N_f > 1$ because two subsequent axial transformations are not an axial transformation.) Finally, the $U_A(1)$ axial transformation is obtained by setting $L = R^\dagger = e^{-ia_0} \mathbf{1}_{N_f}$. The $U_V(1)$ transformation corresponds to $L = R = e^{ia_\pi} \mathbf{1}_{N_f}$, thus trivially implying the identity transformation $\Phi \rightarrow \Phi$.

The effective potential for the field $\Phi$ reads [2]

$$
V_{\text{eff}}[\Phi; \mu^2, \gamma, \delta, k, h] = -\text{Tr}[\mu^2 \Phi^\dagger \Phi + \gamma (\Phi^\dagger \Phi)^2 + \delta (\text{Tr}[\Phi^\dagger \Phi])^2 - k (\text{det} \Phi^\dagger + \text{det} \Phi) - \text{Tr}[h(\Phi^\dagger + \Phi)]].
$$

(3)

The first three terms are invariant upon $U_R(N_f) \times U_L(N_f)$ transformations. A sufficient condition for the stability of the potential is that $\gamma > 0$ and $\delta > 0$. The term proportional to $k$ is not invariant under the $U_A(1)$ axial transformation and describes the so-called axial anomaly. In the last term the diagonal $N_f \times N_f$ matrix $h$ describes the explicit contribution of nonzero current quark masses. It is not invariant under $SU_A(N_f)$ and $U_A(1)$ transformations, and if $h \neq \text{const} \mathbf{1}_{N_f}$, it is also not invariant under $SU_V(N_f)$.

A first, naive attempt to obtain the Mexican hat of (1) is to study the case $N_f = 1$ with $\Phi = \sqrt{\frac{\lambda}{2}} (\sigma + i\tau) = \sqrt{\frac{\lambda}{2}} \vec{\tau}$. In the chiral limit ($h = 0$) one can easily identify $\lambda = (\gamma + \delta)$ and $\mu^2 = - (\gamma + \delta) F^2 < 0$. The latter is a necessary condition for spontaneous symmetry breaking. However, the anomalous term $-k (\text{det} \Phi^\dagger + \text{det} \Phi) = - \sqrt{2} k \sigma$ breaks explicitly chiral symmetry and cannot be regarded as a small perturbation. This is due to the fact that for $N_f = 1$ the chiral transformation $SU_A(N_f)$ cannot be distinguished from the axial transformation $U_A(1)$. We conclude that, in virtue of the anomaly, the Mexican hat potential cannot be reproduced in the case of one quark flavor only.

When $N_f = 2$ the matrix $\Phi$ reads

$$
\Phi = \sum_{a=0}^3 \phi_a \mathbf{1}_a = (\sigma + i\tau) t^0 + (\vec{\sigma}_0 + i\vec{\tau}) \cdot \vec{t},
$$

(4)

where $\vec{t} = \vec{t}/2$, with the vector of Pauli matrices $\vec{t}$, and $t^0 = \mathbf{1}/2$.

In terms of quark degrees of freedom, the scalar isotriplet $\vec{a}_0$ and the pseudoscalar pion $\vec{\tau}$ are given by $\vec{a}_0 \in \mathbf{1}$.

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1In the $N_f = 2$ case one can consider the reduced combination $\Sigma = \sigma^0 + i\vec{\tau}$, which also transforms as $\Sigma \rightarrow R \Sigma L^\dagger$ under chiral $SU_R(2) \times SU_L(2)$ transformation (but not under $U_R(2) \times SU_L(2)$: the axial transformation, which mixes $\Sigma$ with $\vec{\eta}$ and $\vec{\tau}$ with $\vec{a}_0$ cannot be described with $\Sigma$ only). The contributions of the $\gamma$- and $\delta$-terms are equal in this case and the corresponding potential can be written as

$$
V_{\text{MH}} = \frac{1}{2} \{2 \text{Tr}[\Sigma^2 - \Sigma - 2F^2]\}^2 = \frac{1}{2} \{\sigma^2 + \vec{\tau}^2 - F^2\}^2.
$$

Clearly, the model of (1) is recovered by setting $\pi^1 = \pi^2 = 0$. 2The anomaly term proportional to $k$ is in the present form not renormalizable for $N_f \geq 5$. Although this is already in the region of heavy quarks and therefore unimportant for practical purposes, the general issue is if non-renormalizable terms should enter in an effective description of QCD. In principle, being an effective QCD model valid in a restricted low-energy domain, there is no reason to disregard non-renormalizable terms. For instance, Nambu–Jona-Lasinio inspired models of QCD are non-renormalizable. As another example, one can consider the non-renormalizable Fermi Lagrangian of weak interactions, which arises as a low-energy effective term of the more complete (and renormalizable) electroweak Lagrangian. On the other hand, what can constrain the dimensionality of terms entering in an hadronic Lagrangian is rather than the renormalizability—the requirement that dilatation invariance is solely broken by a Yang–Mills scale in the glueball sector; see details in [3] and in Refs. [7–9].